A Mathematical Model Analysis on the Dynamics of Online Game Addiction with Optimal Control

MSc. Thesis

By: Iyasu Kaleb

Hawassa, Ethiopia
June, 2024

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In Partial Fulfillment of the Requirements for the Degree of Masters in Mathematical and Statistical Modeling

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DECLARATION

I hereby declare that **"A Mathematical Model Analysis on the Dynamics of Online Game Addiction with Optimal Control "** is my own work, that it has not been done for any degree in any other universities, and that all the sources I have used have been indicated and acknowledged by complete references.

____________________ ________ ________

Name of Student **Signature** **Date**

APPROVAL PAGE - 1

This is to officially state that the study entitled **"A Mathematical Model Analysis on the Dynamics of Online Game Addiction with Optimal Control"** is an original work carried out by Iyasu Kaleb, ID No. GPMASTR/0004/15 under my guidance and supervision. This is a genuine work that has been done by Iyasu Kaleb for the partial fulfillment of the award of the Degree of Master of Science in Mathematical and Statistical Modeling from Hawassa University. Daily acknowledgments are done during his course of investigation. Therefore, I recommend that it would be accepted as fulfilling the thesis requirements.

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Name of Major Advisor	**Signature**	**Date**

APPROVAL PAGE - 2

We, the undersigned, members of the Board of Examiners of the final open defense by Iyasu Kaleb have read and evaluated his thesis entitled **"A Mathematical Model Analysis on the Dynamics of Online Game Addiction with Optimal Control"** and examined the candidate. This is therefore to certify that the thesis has been accepted in partial fulfillment of the requirement of the Degree of Master of Science in Mathematical and Statistical Modeling.

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SGS Approval	**Signature**	**Date**

Final approval and acceptance of the thesis is contingent upon the submission of the final copy of the thesis to the school of Graduate Studies(SGS) through the Department/School Graduate Committee(DGC/SGC) of the candidate's department.

Stamp of SGS Date :________________

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DEFINITION OF BASIC TERMS

- **Addicted**: Are individuals who constantly play online games.
- **Basic reproduction number:** The expected number of secondary cases generated by a typical addicted individual in a completely susceptible population.
- **Completely recovered**: Are individuals who have completely stopped playing online games after joining the treatment center.
- **Exposed**: Are individuals who play online games occasionally and are at risk of developing addiction.
- **Incompletely recovered**: Are individuals who are partially recovered from playing online games after joining the treatment center.
- **Online game**: An online game is a video game played over the internet where multiple players interact in a virtual environment. They range from simple browser-based games to complex multiplayer games accessible on computers, consoles and mobile phones.
- **Online game addiction**: Online game addiction is excessive and compulsive playing of online video games, leading to negative consequences in various aspects of life. It involves an intense preoccupation with gaming, an inability to control time spent gaming, and the neglect of important responsibilities and activities.
- **Susceptible**: The susceptible are those who have not yet been addicted to the addiction but are likely to get addicted in the future.

LIST OF ABBREVIATIONS

Abbreviation	**Meaning**
EEP	Endemic Equilibrium Point
IA	Internet Addiction
IGO	Internet Gaming Disorder
OC	Optimal Control
OG	Online Game
WHO	World Health Organization

ABSTRACT

Excessive playing of online games leads to addiction, which causes academic underachievement and prevents the daily achievement of goals. Symptoms of addiction are spending more time gaming online and irritability, while risk factors are low self-worth, anxiety and depression. This thesis study focuses on the mathematical model analysis of the dynamics of online game addiction with optimal control. We demonstrated the existence, positivity and boundedness of the solution of the dynamical system. The system has two equilibrium points, namely the online game addiction free and endemic equilibrium points, and their stability was determined using the linearization technique, the Castillo-Chavrz theorem and LaSalles's invariant principle. The basic reproduction number (R_0) of the model was calculated using the principle of the next generation matrix and the sensitivity indices of R_0 to the model parameters were investigated. Additionally, bifurcation analysis was performed to verify the backward and forward bifurcations, and from the analysis, we observed that the model system exhibits forward bifurcation at $R_0 = 1$. The optimal control problem of the model was analyzed using Pontryagin's Maximum Principle and the characterization of the optimal control was constructed. Numerical simulations were performed to verify the accuracy of the analytical results using MATLAB software. From the numerical simulation of sensitivity analysis, decreasing the contact rate with addicted(β_1) and incompletely recovered(β_2) individuals, the addiction rate(δ), the re-addiction rate(ω) and the incomplete recovery rate of treated individuals(τ) decreases the reproduction number. Using optimal control, the combined strategy of minimizing the contact rate with addicted individuals and the re-addiction rate of incompletely recovered individuals minimizes the exposed and addicted individuals as well as the associated cost.

Keywords: *Online game addiction, Treatment, Stability analysis, Sensitivity analysis, Optimal control*

CHAPTER 1
INTRODUCTION

1.1 Background of the study

Online gaming has a long history that begins with some of the earliest mainframe computers. Games were developed to test networks as they were established. Games were created and tested as computers and the internet advanced, from National Aeronautics and Space Administration(NASA) to Massachusetts Institute of Technology(MIT)[40]. A video game that is played mostly online via the internet or similar available computer networks is referred to as an online game[57].

Online games are widely available across a wide range of genres on contemporary gaming platforms, such as PCs, consoles, and mobile devices. Popular genres include strategy games, first-person shooters, and massively multiplayer online role-playing games (MMORPGs). Online games have been operating as games as a service since the 2010s, with revenue models including loot boxes and battle passes acting as downloadable extras on top of free-to-play games. Online games, in contrast to those that can be bought at retail stores, have the drawback of being playable only when specific servers are available. Online gaming has the potential to be both addicting and harmful to a person's health. Moreover, when played excessively, it can lead to inattention, disregard for studies, and even distraction when studying. When people spend too much time in front of computers, gaming can cause mortality[59].

Multiple players play online games at the same time across the globe via the internet or another computer network. Owing to the notable expansion of computer networks, the number of games of this kind is constantly rising. Social networks like Facebook now used for online gaming, which can have both beneficial and bad effects on teenagers. One drawback is that it can lead to addiction, which keeps players from accomplishing their different objectives. In addition, learning and striving for success become problematic when players lack self-control. These games are unhealthy as well. Of the 6 million Indonesians who love online gaming, 40% are teenagers. A survey revealed that among boys and girls aged 12 to 22, 64.45% and 47.85%, respectively, are addicted[63].

The study, conducted at a university in Ethiopia, found that 35.2% of undergraduate students were addicted to the Internet, with 1.8% having severe addiction and 33.4% having mild addiction. Female students had higher addiction rates than males (38.3% vs. 33.4%) and seniors were more addicted than juniors. Factors such as academic performance, parental supervision, self-esteem and peer pressure were linked to Internet addiction. This research suggests the need for interventions to address and prevent Internet addiction among students [43].

Online game addiction has been the subject of numerous studies conducted in Makassar; these studies have included discussions of the social impact of online game addiction despite the application of SIR, SIRS, SEIR, and SEIRS models to COVID-19 cases[63]. Considering the mathematical model for online game addiction as analogous to infectious diseases is crucial for understanding and managing the dynamics of online game addiction. Similarly, it plays a significant role in comprehending addictions such as drug addiction, smoking and social media addiction ([17],[31],[59]). Addiction is a widespread and intricate issue that impacts individuals and societies globally. Therefore, it is essential to comprehend the underlying mechanisms and dynamics of addiction to develop effective prevention and treatment strategies. The mathematical model serves as a powerful tool for studying addiction, providing insights into the underlying processes, predicting outcomes and guiding decision-making.

Video game addiction is becoming a more significant problem in today's society and causes disorders. Notably, the recognition of gaming disorder as an official mental health condition is directly associated with the use of digital technology [58]. The WHO has officially classified game addiction as a mental health disorder in the 11^{th} edition of the International Classification of Diseases[59]. This recognition emphasizes the seriousness of the problem and the importance of treating and knowing how excessive gaming affects people's mental health.

According to a 2021 systematic review and meta-analysis, there are already over two billion gamers, and by 2023, that figure is probably going to rise to three billion[46]. Addiction to video games affects 3%–4% of gamers, and the prevalence globally has been found to be 3.05%, which implies that the number of people with gaming problems may reach 60 million or higher[46].

Although the concept of Internet Gaming Disorder (IGD) has been questioned, the WHO recognized internet gaming addiction as a real disorder[23, 50, 49]. Gaming disorder, predominantly online, is now commonly mentioned as Internet Gaming Disorder (IGD), which is generally defined as ”Persistent and recurrent use of the internet to engage in games, often with other players, leading to clinically significant impairment or distress.” The Diagnostic and Statistical Manual of Mental Disorders (DSM-5) outlines criteria for classifying Internet Gaming Disorder (IGD), though it's not intended for clinical diagnosis[55]. These criteria include:

- Internet game obsession taking over daily activities;
- Symptoms of withdrawal (such as irritability, anxiety, or sadness without any outward manifestations of pharmacological withdrawal);
- Lack of control over internet gaming;
- Requirement to spend more time gaming online;

- Interest in past pastimes and entertainment dwindles;
- Excessive use persists despite awareness of psychological issues;
- Falsifying information to family, therapists, or other people regarding the quantity of online gaming;
- Running away from or relieving a bad mood (such as helplessness, guilt, or anxiety);
- Losing major relationships, jobs, or chances for education or employment.

While these criteria highlight the potential for negative impacts, it is important to note that video game players often form communities, join groups, become friends, support each other and gain social standing, similar to athletes. All of this can have a virtual context, but as time goes on, the connection becomes actual. The majority of players desire to feel important and to be a part of something, so they typically play a specific role. This frequently happens when someone lacks social fulfillment; as a result, playing video games takes center stage in their social lives and boosts their self-esteem. The following are typical risk factors for video game addiction[75]: Low self-worth, Social isolation, Anxiety, Neglect-related feelings, Empathy deficit, Depression, An antagonistic disposition and Mood management with video games.

We know the usefulness of technology in education as well as its impact on the younger generations[74]. The prevalence of video games has increased over the years, capturing many people's imaginations worldwide since the 1970s. The graphics of older video games were simpler than today's multifaceted games, which have thus attracted many children and adults [16].

However, video gamers waste a lot of time trying to win. Video game addiction has been identified as a result of increased gaming. Advancements in technology and gaming equipment have made video games easily accessible. Additionally, the prevalence of handheld gaming devices has contributed to video game addiction. Video game addiction has negative effects on gamers despite being a controversial concept. Video game addiction is a serious issue that requires treatment. So, to treat video game addiction educational and rehabilitation strategies have been used[75].

1.2 Statement of the problem

Online game addiction has become a growing concern in recent years, with detrimental impacts on teenagers' lives in a number of ways, including increased psychoticism, depression and anxiety, as well as family relationship problems, lower quality of life, increased social anxiety, academic underachievement and improved sleep deprivation [24]. Hence, it seems to be quite significant to undertake more research on the spread dynamics of addiction and a serious control strategy must be needed. Saman et al. [59] studied the mathematical model and optimal control of online game addiction. However, the individual's commitment to quitting and providing effective treatment were not considered.

We modify the model, considering the individual's commitment during the time of treatment and the effectiveness of the treatment, which leads to individuals recovering completely and temporarily. Also, susceptible groups of individuals become exposed due to contact with temporarily recovered individuals. Furthermore, to minimize the negative consequences of excessive gaming, such as socio-economic costs (the financial burden on individuals, families, etc.) and the impact on the individual's well-being, various interventions can be explored, such as educational campaigns, parental guidance, time restrictions and psychological support. Such strategies of intervention help policymakers and healthcare professionals minimize the number of addicted individuals in the population and provide effective treatment.

Thus, the present work is a modification of [59] by incorporating treated, completely recovered and temporarily recovered individuals. The study is expected to address the following basic questions:

- How to modify a mathematical model for the dynamics of online game addiction?
- How the positivity and boundedness of solutions, equilibria and basic reproduction number of the model are established?
- How to demonstrate the stability of the equilibria?
- How to minimize the spread of OG addiction with optimum implementation cost?

1.3 Objectives of the study

1.3.1 General objective

The general objective of this study is to formulate and analyze a mathematical model of online game addiction with optimal control.

1.3.2 Specific objective

The specific objectives of the study are:

- To formulate a modified mathematical model for transmission dynamics of online game addiction.
- To show the existence, positivity and boundedness of the solution of the model.
- To compute the basic reproduction number of the model.
- To establish the existence, stabilities of the equilibrium points and sensitivity analysis of the model.
- To optimize the rate of contact with the addicted and re-addiction rate of the model.
- To perform the numerical simulation and analyze it using MATLAB software.

1.4 Significant of the study

In today's digital age, online gaming addiction has become a widespread problem, affecting people all over the world. It is no longer limited to personal struggles but has grown into a global issue[52, 71]. Thus, the findings of this study have great importance to stakeholders (policy makers, healthcare professionals, the community, etc.) in understanding the underlying mechanisms as well as in preventing and controlling addiction through treatment and creating awareness about the impact and ill-effects. It helps in the identification of the factors that contribute to the development of addiction. It can be useful for policymakers and healthcare professionals to better understand the complex dynamics of this phenomenon, create further awareness, and provide effective treatment. Furthermore, the optimal control strategies in this study can help assess the economic and social impacts of online game addiction while increasing understanding of addiction dynamics and providing valuable insights for designing effective strategies to minimize the impact of OG addiction with minimum cost.

1.5 Scope of the study

This thesis mainly focuses on the study of the transmission dynamics of online game addiction with optimal control strategies. A mathematical model of online gaming addiction that is formulated and analyzed in this study can identify risk factors and inform prevention and intervention strategies. It can also be used to predict addiction and compare the efficacy of various treatments. Additionally, it can help researchers understand the underlying mechanisms of addiction and guide future research directions.

1.6 Limitation of the study

The first limitation of this study is that we did not collect real data due to the time limitation. Instead, we utilized parameter values from the literature; some are assumptions used for simulation purposes. The second issue with this study is that it may not fully capture the complexity and individual variability of human behavior. Addiction is a multifaceted and nuanced phenomenon influenced by various psychological, social and environmental factors, so the model may oversimplify the complexities of addiction and may not fully account for the unique experiences and circumstances of each individual struggling with online game addiction. Additionally, the model may not encompass the full range of potential consequences and impacts of addiction on mental health, relationships and overall well-being. Furthermore, the outcomes of this thesis may not be substantially utilized as it was conducted without actual data and without a designated case study area.

1.7 Organization of the thesis

The thesis is organized as follows: Chapter (1) is the introduction, which comprises the background of the study, statement of the problem, objectives, and significance of the study. Chapter (2) presents a review of the related literature. Chapter (3) presents some epidemiological preliminaries and the methodology. The mathematical model is formulated and described in Chapter (4). The qualitative analysis of the modified model by examining the equilibrium points and its stability analysis is studied in this chapter, along with numerical simulations to augment the accuracy of the analytical results. In Chapter (5) , an optimal control problem is formulated and studied using Pontryagin's Maximum Principle. In this chapter, numerical simulations are used for optimality systems. Conclusions and recommendations from the study are given in Chapter (6).

CHAPTER 2
LITERATURE REVIEW

Because of the global COVID-19 pandemic in 2020, some people who had dropped out of online games became re-addicted to them due to the stay-at-home orders, exacerbating the already existing phenomenon of online game addiction. Controlling the prevalence of online game addiction was of great significance in protecting people's healthy lives. To address this, researchers Li and Guo [33] developed a mathematical model of online game addiction with incomplete recovery and relapse, incorporating a group of professionals who work in professional games or games-related jobs, including vulnerable, infectious, professional and quitting compartments. They divided the total population into susceptible,exposed, professional, incompletely recovered, and completely recovered quitters and considered it to represent a direct transfer from susceptible people to professionals. They used data of e-sports users in China from 2010 to 2020. Their findings suggest that the spread of game addiction in China was very serious since the basic reproduction number they obtained was $R_0 \approx 5.05$. They suggested that by increasing continuous attention to incompletely recovered people, we could prevent more people from becoming addicted to games.

Saman et al.[59] developed the SEIR model in the same way as the mathematical model of infectious disease prevalence, classifying online game addiction as a social disease. A review of mathematical models for the spread of infectious diseases, including dengue fever, hepatitis, tuberculosis and COVID-19. They studied the transmission dynamics of online game addiction, considering a susceptible, exposed, infected and recovered population of Makassar City junior high school children. They also identified optimal control strategies through guidance and counseling for students addicted to online games, as well as analyzed and simulated models to predict the proportion of students who manage their online game addiction and those who do not.

Saman et al.[59] study finding show the basic reproduction number (R_0) dropped from 0.2221 to 0.1342 after control, indicating students can overcome online gaming addiction with guidance and counseling. Consequently, they suggested that the SEIR model can serve as a guide for the best possible control to recognize addictions to online games, and junior high school students who become addicted to online gaming can be minimized, while the number of students who give up on the practice can be increased with ideal control through guidance and counseling.

Li and Guo[31] studied the transmission dynamics of online game addiction with the populations susceptible, infected, professional and quitting. They considered the population on mainland China and the time individuals spent per day with a job as the following: Susceptible individuals whose gaming time was less than 5 hours per day, infected individuals whose gaming time was more than 5 hours per day and who lacked a proper job, professional individuals whose gaming time was more than 5 hours per day and who had a proper job, quitting individuals who quit playing the game. They considered the direct transfer from susceptible individuals to professional individuals.

According to the study done by Li and Guo [31], there might be a direct transfer of risk from vulnerable individuals to professionals. After formulating the model, they investigated the basic reproduction number (R_0) of the model using the next-generation matrix method. Based on R_0, the stability of all kinds of equilibria was obtained. They used the Lyapunov function to study the global stability of the two equilibria. Furthermore, two time-dependent control variables are taken and a control strategy is taken to limit the proportion of the susceptible individual who contacts the infective individual and limit the infective individual's turn into other compartments. Their findings suggest that the two control forces (prevention and treatment) needed to be quickly elevated to their maximum in the early stages of an excess epidemic. It is possible to reduce the maximum control intensity to some degree once the situation has improved, and to reduce treatment costs, the best control approach may effectively prevent the emergence of addiction.

Maulana [38] studied the dynamics of online game addiction. The study employed the SEAR model, which consisted of four compartments: susceptible, exposed, addiction and recovery. Dimas Avian Maulana used the primary data obtained from a survey in the Faculty of Mathematics and Natural Sciences, Universitas Negeri Surabaya. Dimas Avian Maulana focused solely on individuals addicted to online games on gadgets and did not consider the impact of professional players (e-sports). The study's two equilibria have been established. As a result of his finding, concluded that there is no widespread addiction to internet gaming since they obtained $R_0 = 0.836$ and the sensitive parameters of the model are the rate of susceptible gamers being exposed and the rate of individuals who are addicted to online games.

Guo and Li[17] presented a mathematical model considering the family education factor and they divided the population into six compartments based on family education and the time they spent. They carried out a qualitative and dynamic analysis. They examined the existence and stability of equilibria, the basic reproduction number (R_0), and the non-negativity and boundedness of solutions. They used control methods to determine the best possible control expression model that incorporates family education, isolation, and treatment. As their findings suggest, the lack of family education has a great impact on teenagers' game addiction. And implementing family education programs can prevent more teenagers from developing a gaming addiction and more successfully manage the epidemic of game addiction.

The initial model for our study was developed by Saman et al.[59]. We extended the model, considering the individual's commitment during the time of treatment and the effectiveness of the treatment, which leads to individuals recovering completely and temporarily. In addition to that, susceptible groups of individuals become exposed due to contact with temporally recovered individuals, not solely with addicted individuals; due to escapism, contact with addicted individuals and dopamine release temporary recovered individuals re-addicted.

The SEIR model, developed by Saman et al.[59], is commonly used to study the spread and dynamics of infectious diseases. When applied to online game addiction, it assumes that individuals transition from being susceptible to becoming exposed, infected and eventually recovered. However, this model does not account for the potential impact of treatment interventions or the possibility of an incomplete recovery. Incorporating treatment and incomplete recovery classes into the SEIR model, considering the effectiveness of different treatment approaches (e.g., cognitive-behavioral therapy, medication, support groups), the duration of treatment and the rate of relapse, allows us to better understand the dynamics of online game addiction and the potential impact of interventions on the spread and recovery from this addiction.

By addressing this gap, the Susceptible-Exposed-Addicted-Treated-Incompletely recovered-Completely recovered(SEATQ$_i$Q$_p$) model provides valuable insights into the effectiveness of different treatment approaches for online game addiction and informs the development of evidence-based interventions. Additionally, understanding the dynamics of incomplete recovery can help identify factors that contribute to relapse and inform the design of interventions aimed at promoting sustained recovery and preventing relapse in individuals with online game addiction. Thus, this study is intended to fill that gap by developing and analyzing a mathematical model of online game addiction incorporating the individual's commitment and effective treatment effect. Also, both the analytical and numerical studies of the model are conducted to obtain necessary information that could be useful towards reducing the spread of addiction. Therefore, this study will help us understand the extent of online game addiction and the effects of commitment and effective treatment.

CHAPTER 3
METHODOLOGY

3.1 Preliminaries

In this chapter, we present a methodological approach that will be used to analyze the OG addiction model. A mathematical model that we developed made use of nonlinear ordinary differential equations and a deterministic compartmental model. We investigate the equilibrium that is addiction-free and the stability and well-posedness of the model. We compute the basic reproduction number using the next-generation matrix and analyze the local and global stability of equilibrium points. We use a sensitivity analysis to identify key variables that contribute to the spread of addiction and employ an optimal control strategy to lower the number of addicts. To simulate the model, MATLAB software is used.

3.2 Autonomous system of ordinary differential equations

Suppose we have a n-dimensional autonomous system of the following type:

$$\begin{aligned} \hat{x} &= f(x(t)), \\ x(t_0) &= x_0, \end{aligned} \tag{3.2.1}$$

where $x_0, x \in D \subset \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$; with f is continuous at $x \in D \subset \mathbb{R}^n$.

Definition 3.2.1. [81](**Well-posedness**)
An initial value problem(IVP) given in (3.2.1) is mathematically well-posed if and only if the following conditions hold:

i. Its solution exists,

ii. Its solution is unique,

iii. Its solution continuously depends on an initial condition.

As we are working with populations and consider epidemiological models for OG addiction the next two prerequisites are crucial:

1. The solution should be non-negative over time,
2. The solution should be bounded.

Theorem 3.2.1. [41](**Picard's theorem**)
Consider the initial value problem given in (3.2.1). If the function f is continuous and that all its partial derivative $\frac{\partial f_i}{\partial x_j}$, for $i, j = 1, 2, 3, ...n$ are continuous for x in some open connected set $D \subset \mathbb{R}^n$, then for $x_0 \in D$ the problem (3.2.1) has a solution $x(t)$ on the some time interval $(-\tau, \tau)$, $\tau > 0$ about $t = 0$, and the solution is unique.

Theorem 3.2.2. [65](**The uniqueness of solution**)
Let $\mathbb{D}$ denotes the region defined:

$$|t - t_0| \leq k, ||x - x_0|| \leq n, x = (x_1, x_2, ..., x_6), x_0 = (x_{01}, x_{02}, ..., x_{0n}) \tag{3.2.2}$$

and suppose that $f(t, x)$ satisfies the Lipschitz condition

$$||f(t, x_1) - f(t, x_2)|| \leq p||x_1 - x_2||, \tag{3.2.3}$$

when the pairs (t, x_1) and (t, x_2) belongs to $\mathbb{D}$ where p is a positive constant. Then, there exists a constant $\eta > 0$. Such that there exists a exactly one continuous vector solution $x(t)$ of the system(4.2.1) in the interval $|t - t_0| < \eta$. With the condition (3.2.2) $\frac{\partial f_i}{\partial x_j}, i, j = 1, 2, ..., 6$ is continuous and bounded in $\mathbb{D}$.

Lemma 3.2.1. [65] If $f(t, x)$ has continuous partial derivative $\frac{\partial f_i}{\partial x_j}$ on a bounded closed convex domain $\mathbb{R}$(convex set of real numbers), where $\mathbb{R}$ is used to denotes real numbers, then it satisfies a Lipschitz condition in $\mathbb{R}$. In the domain

$$1 \leq \epsilon \leq \mathbb{R}. \tag{3.2.4}$$

So, we look for a bounded solution of the form

$$0 < \mathbb{R} < \infty.$$

Definition 3.2.2. [76](**Positivity of solution**)
The solution of given autonomous system, of (3.2.1) is said to be positive, if all trajectories $x(t)$ positive for any $t \geq 0$.

Definition 3.2.3. [53] (**Boundedness of solution**)
The positive solution given in (3.2.6) autonomous system, of (3.2.1) is said to be bounded, if any solution, of $x(t, t_0, x_0)$ (3.2.1) satisfies

$$||x(t, t_0, x_0)|| \leq C(||x_0||, t_0)$$

for all $t \geq t_0$ where, $C : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a constant that depend on t_0 and x_0.

Theorem 3.2.3. [27] Let a linear ordinary differential equation is given by

$$\dot{y}(x) + p(x)y = q(x). \tag{3.2.5}$$

Then using integrating factor method the integrating factor and its general solution, respectively, are given by

$$IF(x) = e^{\int_{x_0}^{x} p(s)ds} \tag{3.2.6a}$$

$$y(x) = e^{-\int_{x_0}^{x} p(s)ds}(C + \int_{x_0}^{x} q(x)e^{\int_{x_0}^{x} p(s)ds}dx), \tag{3.2.6b}$$

where C is an arbitrary constant of integration.

Theorem 3.2.4. [3] Let A be a constant matrix. The solution of the system $\hat{x} = Ax + f(t)$, with initial conditions $x(t_0) = x_0$, is given by

$$x(t) = \phi(t)\phi(t)^{-1}x_0 + \int_{t_0}^{t} \phi(t - s + t_0)\phi(t_0)f(s)ds, \tag{3.2.7}$$

$\phi(t)$ is any fundamental matrix for the system $\phi = A\phi$.

3.2.1 Stability analysis of equilibrium points

Definition 3.2.4. [48] Given the autonomous system (3.2.1), a state x^* is said to be an **equilibrium point** of the system (3.2.1) if $f(x^*) = 0$.

Definition 3.2.5. [39] The solution x^* is said to be **stable** if for every $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ such that, $|x^* - x_0| < \delta \implies |x^* - x(t)| < \epsilon, t > t_0, t_0 \in \mathbb{R}$, for every solution $x(t)$ of (3.2.1) with $x(0) = x_0$.

Definition 3.2.6. [39] An equilibrium point x^* is attracting if there is a $\delta > 0$ such that $|x_0 - x^*| < \delta \implies x(t) \to x^* \text{as} t \to \infty$, for every solution $x(t)$ of (3.2.1) with $x(0) = x_0$. (All trajectories that start near x^* approach it as $t \to \infty$.)

Definition 3.2.7. [39] An equilibrium point x^* is asymptotically sable if it is stable and attracting. In other words, solution x^* is said to be asymptotically-stable if:

i. it is stable, and

ii. there exists a constant $\delta > 0$ such that for any solution $x(t)$ of (3.2.1) satisfying $|x^* - x(0)| < \delta$, then $\lim\limits_{t\to\infty} |x(t) - x^*| = 0$

Definition 3.2.8. [39] An equilibrium point x^* is said to be globally asymptotically stable if it is asymptotically stable for initial condition $x_0 \in \mathbb{R}^n$.

Definition 3.2.9. [39] An equilibrium point x^* of the model (3.2.1) is said to be **locally asymptotically stable** if it is locally stable and every trajectory that starts sufficiently close to **x** $\to x^*$ as $t \to \infty$. A study state x^* which is not stable is said to be unstable.

Definition 3.2.10. [39] An equilibrium point of a given dynamics system is **stable** means all solution curves of the equation **attracts towards the equilibrium point**, while unstable means all solution cures of the dynamic system go away from the equilibrium point.

Definition 3.2.11. [48] An equilibrium point x^* is **globally stable** if all trajectories converges to x^* i.e $\lim\limits_{t\to\infty} x(t) = x^*$.

Proposition 3.2.1. [39] A set M is said to be an **invariant set** with respect to the autonomous ordinary differential equation (3.2.1) if $x(0) \in M \implies x(t) \in M; \forall t \in \mathbb{R}$, i.e., if any trajectory starts in M it will stay in M $\forall t$. A set M is a positively invariant set with respect to (3.2.1) if $x(0) \in M \implies x(t) \in M; \forall t \geq 0$.

Definition 3.2.12. [48] A function V:$\mathbb{R}^n \to \mathbb{R}$ **is positive definite** if

i. $V(x) > 0 \, \forall x \neq x^*$,

ii. $V(x^*) = 0$ if and only if $x = x^*$,where x^* is an equilibrium of (3.2.1).

Local stability using linearization

Mathematically, the linearized system at the equilibrium point can be utilized to analyze the stability of equilibrium.

Theorem 3.2.5. [72] The Jacobian matrix associated to the system (3.2.1) at the equilibrium point x^*, which is denoted by $Df(x^*)$, is given by the matrix

$$Df(x^*) = \left[\frac{\partial f_i(x^*)}{\partial x_j}\right] = \begin{bmatrix} \frac{\partial f_1(x^*)}{\partial x_1} & \frac{\partial f_1(x^*)}{\partial x_2} & \cdots & \frac{\partial f_1(x^*)}{\partial x_n} \\ \frac{\partial f_2(x^*)}{\partial x_1} & \frac{\partial f_2(x^*)}{\partial x_2} & \cdots & \frac{\partial f_2(x^*)}{\partial x_n} \\ \vdots & & & \vdots \\ \frac{\partial f_n(x^*)}{\partial x_1} & \frac{\partial f_n(x^*)}{\partial x_2} & \cdots & \frac{\partial f_n(x^*)}{\partial x_n} \end{bmatrix}_{|x=x^*}$$

where $i, j = 1, 2, ...n$.

Proposition 3.2.2. [48] An equilibrium point x^* of the dynamical system (3.2.1) is **locally asymptotically stable** if all eigenvalues of the Jacobian

$$Df(x^*) = \left[\frac{\partial f_i(x^*)}{\partial x_j}\right]$$

evaluated at x^* are negative. The equilibrium x^* is unstable if at least one of the eigenvalues of $Df(x^*)$ is positive.

3.2.2 Routh-Hurwitz stability criterion

A necessary and sufficient condition for each root of the characteristic polynomial with real coefficients is provided by the Roth Hurwitz criterion, which makes it significant. A non-linear system of differential equations' asymptotic stability at an equilibrium point can be determined using the Routh-Hurwitz criterion. Let us examine the characteristic equation of degree n, as provided by

$$Q(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_{n-1}\lambda + a_n,$$

where a_i is the real constant for all polynomial coefficients for $i = 1, 2, ...n$. By using the coefficients a_i of the characteristics polynomial, define the $n \times n$ Hurwitz matrix.

$$H_1 = [a_1], H_2 = \begin{bmatrix} a_1 & 1 \\ a_3 & a_2 \end{bmatrix}, \cdots, H_n = \begin{bmatrix} a_1 & 1 & 0 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & 1 & \cdots & 0 \\ a_5 & a_4 & a_3 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_n \end{bmatrix},$$

The Routh Hurwitz criteria for polynomials of degrees $n = 2, 3, 4$ and 5 can be summed up as follows:

$n = 2 : a_1 > 0; a_2 > 0$ and $a_1a_2 > 0$

$n = 3 : a_1 > 0; a_3 > 0$ and $a_1a_2 > a_3$

$n = 4 : a_1 > 0; a_3 > 0; a_4 > 0$ and $a_1a_2a_3 > a_3^2 + a_1^2a_4$

$n = 5 : a_1 > 0$ for $i = 1, 2, 3, 4, 5; a_1a_2a_3 > a_3^2 + a_1^2a_4$ and

$$(a_1a_4 - a_5)(a_1a_2a_3 - a_3^2 - a_1^2a_4) > a_5(a_1a_2 - a_3)^2 + a_1a_6^2.$$

Global stability

The two well-known findings that are utilized in epidemiological modeling to global stability will be presented here.

Theorem 3.2.6. [28](**Lyapunov Stability Theorem**)
Let x^* be an equilibrium solution of (3.2.1) and suppose that we can find a Lyapunov function i.e a continuously differentiable, real and valued function $V(x)$ such that

i. $V(x) > 0 \ \forall x \neq x^*$, and $V(x^*) = 0$. (V is positive definite),

ii. $\frac{dV(x)}{dt} < 0 \, \forall x \neq x^*$.

Then x^* is globally asymptotically stable; for all initial conditions, $x(t) \to x^*$ as $t \to \infty$.

Theorem 3.2.7. [5] (**Castillo Chavez Theorem**)
Assume that the system (3.2.1) can be rewritten in the form

$$\frac{dZ_1}{dt} = F(Z_1, Z_2) \tag{3.2.8a}$$

$$\frac{dZ_2}{dt} = G(Z_1, Z_2) \tag{3.2.8b}$$

where the $Z_1 \in \mathbb{R}$(represents the classes of non-infected individuals) and $Z_2 \in \mathbb{R}$(represents the classes of infected individuals). Assume that $G(Z_1, 0) = 0$ and let $E^0 = (Z_1^*, 0)$ be a steady state of (3.2.1)(the disease free equilibrium point). If the following conditions are satisfied:

I. For the system $\frac{dZ_1}{dt} = F(Z_1, 0)$, the steady state Z_1^* globally asymptotically stable.

II. $G(Z_1, Z_2) = AZ_2 - \hat{G}(Z_1, Z_2).\hat{G}(Z_1, Z_2) \geq 0$ for $(Z_1, Z_2) \in \Omega$,where A is a Metzler matrix(the off diagonal elements of A are non-negative) and Ω is the region where the model makes biological sense.
Then the steady sate $E^0 = (Z_1^*, 0)$ is globally asymptotically stable for the system (3.2.8) provided that the basic reproduction number of the model is less than one.

3.3 Basic concepts in game addiction modeling as epidemiological modeling

Definition 3.3.1. [45] Epidemiology is the study of the occurrence and consequences of health related conditions or occurrences in certain groups, as well as the application of this study to the prevention and management of health issues.

3.3.1 Online game addiction free equilibrium point

When a population is impacted by a virtual epidemic of game addiction, it is possible for the epidemic to be completely eradicated from the population. In epidemiological modeling, an OG addiction-free equilibrium represents a stable state where all coordinates in the game addiction compartments are reduced to zero. This condition is known as the population being online game addiction-free in the game world. When online game addiction-free equilibrium is stable, we expect that the population with game addiction will be OG addiction-free as time progresses.

3.3.2 Endemic equilibrium point (EE)

Unlike the OG addiction-free equilibrium, an endemic equilibrium point is a steady state in which at least one of its coordinates in the game addiction compartment is non-zero. It is a steady-state solution where OG addiction persists in the population.

3.3.3 The basic reproduction number

Because of its significance and critical importance, the basic reproduction number idea is one of the main subjects of mathematical modeling of infectious diseases. This number is almost always mentioned in publications on mathematical models. As it gives an understanding of the disease's future condition in the population. Implications: Since we are studying online game addiction modeling, it is crucial and informs us about the probability of the addiction persisting or eradicating.

Definition 3.3.2. [5]. The basic reproduction number, denoted by R_0, is defined as the average number of secondary infections that are produced when a single infected individual is introduced into a population of purely susceptible individuals.

If $R_0 > 1$, then on average, each single addicted individual can addict more than one person. Hence, the total number of addicts will continuously increase, and the addiction will become endemic (it will remain in the population forever). However, when $R_0 < 1$, on average, each single addicted individual can addict zero or at most one person. Over time, the number of newly addicted people will decrease, and the population can become online game addiction-free. Therefore, the OG addiction-free equilibrium is stable when $R_0 < 1$ and unstable when $R_0 > 1$.

The basic reproduction number R_0 is computed using the method of the next-generation matrix. Assume there are n online gamer compartments and m non-online gamer compartments. Let $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ be sub-populations in each compartment. Further denote by F_i the rate of secondary infection increase in the i^{th} compartment and V_i the online game addiction progression rate, death, and recovery decrease in the i^{th} compartment. The compartment model can be written as:

$$\begin{aligned} \frac{dx_i}{dt} &= F_i - V_i(x,y), i = 1,2,3,...,n \\ \frac{dy_i}{dt} &= G_j(x,y), j = 1,2,3,...,m \end{aligned}$$

where

$$\begin{aligned} F &= (\frac{\partial F_i}{\partial x_j}(0, y^*))_{1 \le i,j \le n} \\ V &= (\frac{\partial V_i}{\partial x_j}(0, y^*))_{1 \le i,j \le n} \end{aligned}$$

then FV^{-1} is called the next generation matrix. The basic reproduction number is the spectral radius (dominant eigenvalue) of the matrix FV^{-1} that is, $R_0 = \rho(FV^{-1})$.

3.3.4 Center manifold theory

Center manifold theory has been used to determine the local stability of a non-hyperbolic equilibrium (the linearization matrix has at least one eigenvalue with a zero real part) [10]. We shall describe a theory that not only can determine the local stability of the non-hyperbolic equilibrium but also t settles the question of the existence of another equilibrium (bifurcated from the non-hyperbolic equilibrium). This theory is based on general center manifold theory.

Theorem 3.3.1. [6] Consider a general system of ODEs with parameters ϕ:

$$\frac{dx}{dt} = f(x, \phi), f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \quad and \quad f \in \mathbb{C}^2(\mathbb{R}^n \times \mathbb{R}). \tag{3.3.1}$$

It is assumed that 0 is an equilibrium for the system (3.3.1) for all values of the parameters ϕ, i.e.

$$f(0, \phi) = 0 \quad \forall \phi. \tag{3.3.2}$$

Assume that:

A1. A=$D_x f(0,0) = [\frac{\partial f_x}{\partial x_j}(0,0)]$ is the linearization matrix of system (3.3.1) around the equilibrium 0 with ϕ evaluated at 0. Zero is a simple eigenvalue of A and all other eigenvalues of A have negative real part;

A2. Matrix A has a non-negative right eigenvector w and a left eigenvector v corresponding to the zero eigenvalue.

Let f_k be the k^{th} component of f and

$$a = \sum_{k,i,j=1}^{n} v_k w_i w_j \frac{\partial^2 f_k}{\partial x_i \partial x_j}(0,0), \tag{3.3.3a}$$

$$b = \sum_{k,i=1}^{n} v_k w_i \frac{\partial^2 f_k}{\partial x_i \partial \phi}(0,0). \tag{3.3.3b}$$

The local dynamics of the system (3.3.1) around 0 is totally determined by the sings of a and b;

i. a$>$ 0, b$>$0. When $\phi < 0$ with $|\phi|<<1$, 0 is locally asymptotically stable, and there exists a positive unstable equilibrium; when $0 < \phi<<1$, 0 is unstable and there exists a negative and locally asymptotically stable equilibrium;

ii. $a < 0, b < 0$. When $\phi < 0$ with $|\phi| << 1$, 0 is unstable; when $0 < \phi << 1$, 0 is locally asymptotically stable, and there exists a positive unstable equilibrium;

iii. $a > 0, b < 0$. When $\phi < 0$ with $|\phi| << 1$, 0 is unstable, and there exists a locally asymptotically stable negative equilibrium; when $0 < \phi << 1$, 0 is stable, and a positive unstable equilibrium appears;

iv. $a < 0, b > 0$. When ϕ changes from negative to positive, 0 changes its stability from stable to unstable. Correspondingly a negative unstable equilibrium becomes positive and locally asymptotically stable.

Specifically, forward bifurcation occurs at $\phi = 0$ if $a < 0, b > 0$; a backward bifurcation occurs at $\phi = 0$ if $a > 0, b > 0$.

3.3.5 Bifurcation in OG addiction models

Definition 3.3.3. For a dynamical system(3.2.1), a qualitative change in its dynamics created by changing parameters is known as bifurcation. The point of the parameter where the change occur is called bifurcation point.

Bifurcation occurs when a parameter change affects the stability of an equilibrium point number or type of equilibria. When the value of R_0 increases from $R_0 < 1$ to $R_0 > 1$, behaviors of the solution to the system of equations in the feasible region undergo qualitative changes in both the number or type of equilibria and their stability. We say that the system undergoes a bifurcation, and $R_0 = 1$ is the bifurcation value.

In epidemiological model, most of the time there are two bifurcations at $R_0 = 1$ namely: forward and backward bifurcation. A forward bifurcation exist when R_0 crosses unity from below; a small positive asymptotically stable equilibrium occurs and the OG addiction free-equilibrium point losses its stability. Conversely, a backward bifurcation exist when R_0 less than unity; a small positive unstable equilibrium occurs while the OG addiction free-equilibrium and a larger positive equilibrium are locally asymptotically stability. Epidemiologically, some times a backward bifurcation occurs when $R_0 < 1$, which implies that the OG addiction-free point and the endemic equilibrium point co-exist. In this case, only reducing R_0 to less than unity is not enough to eliminate addiction.

3.3.6 Sensitivity analysis

Sensitivity analysis is used to examine how a change in the value of the predictor of the model parameter affects the system's dynamic behavior. The analysis helps in finding the degree to which input parameters and prediction parameters are connected, which is a helpful technique. It also helps in determining the degree of change an input parameter must undergo in order for a predictor parameter to take on the intended value. So, to change the model's output, it needs to change the values of the most sensitive parameters appropriately.

Definition 3.3.4. For a certain quantity R, the normalized forward sensitivity index with respect to a parameter p is obtained as the following:

$$\Gamma_p^R = \frac{\partial R}{\partial p} \times \frac{p}{R}.$$

3.4 Optimal control theory

An extension of several variables in the calculus of variations, optimal control theory was built on the groundbreaking work of Lev Pontryagin and Richard Bellman in the 1950s, which came after Edward J. McShane. It is a mathematical optimization technique for determining control policies. The theory is a powerful mathematical tool that is applied to decision-making in complicated biological contexts involves determining how best to distribute funds over time in order to reduce the negative impacts of addiction and the expenses related to bringing these methods into reality.

The theory is useful in developing ideas to consider limitations like resource availability and the dynamics of OG addiction progression and effectively balance various treatments over time to result in the greatest outcomes of interventions, such as counseling services or educational programs, in order to reduce the impact of online game addiction. Guaranteeing the achievement of particular optimality requirements is addressed by optimal control theory. Creating control rules that restrict access to gaming platforms and establish time limitations for gaming sessions may be necessary in the case of online game addiction. Additionally, OC provides an approach to finding out the state trajectories and optimal control for a dynamic system over a given amount of time [56]. To customize interventions, strategies and policies that attempt to reduce the impact of online gaming and encourage healthier gaming behaviors and general well-being by using OC.

Optimal control theory has different procedures and/or ways of reaching the optimality of the deserved problem. To describe it, let us consider the following controlled dynamical system.

$$\begin{aligned} &x'(t) = g(t, x(t), u(t)),\ t \in [t_0, t_f] \\ &x(t_0) = x_0,\ x(t_f) = x_{t_f}. \end{aligned} \tag{3.4.1}$$

Within the dynamic system (3.4.1), the state variable at a given time t is denoted by $x(t)$, the initial condition of the state variables is denoted by x_0, the final condition of the state variables is denoted by x_{t_f}, the terminal innervation time is denoted by t_f and $u(t)$ denotes the time-dependent control parameter. The main objective is to optimize cost-function by controlling the dynamical system.

3.4.1 The cost function

A mathematical equation describing the production output corresponds to the optimization of the target with respect to the optimal problem and the initial condition is the cost function such that:

$$
\begin{aligned}
&\text{maximize} \quad J(t,x,u) = \int_{t_0}^{t_f} f(t,x(t),u(t))dt, \\
&\text{subject to} \quad x'(t) = g(t,x(t),u(t)),
\end{aligned} \tag{3.4.2}
$$

where $x \in X \subset \mathbb{R}^n$, with initial and terminal conditions in (3.4.1). A control set is a set of points characterized by $u(t) \in U \subset \mathbb{R}^m, m \in \mathbb{N}$. A control variable $u(t)$ is said to be an admissible control if it is piecewise continuous defined on some time interval $t_0 \leq t \leq t_f$ with range in the control region U, $u(t) \in U, \forall t \in [t_0,\ t_f]$[29].

3.4.2 Pontryagin's Maximum Principle(PMP)

Pontryagin's Maximum Principle states a necessary condition that must hold on an optimal trajectory. An optimal control problem consists of finding a piecewise continuous control $u(t)$ and the associated state $x(t)$ that optimizes a cost functional $J[x(t), u(t)]$. Many mathematical models that use optimal control theory rely on Pontryagin's Maximum Principle, which is a first-order condition for finding the optimal solution. For convenience, this is discussed below.

Theorem 3.4.1. [29] (**Ponryagin's Maximum Principle(PMP)**)
Let $u(t)$ be a time optimal control and $X(t)$ be the corresponding response system. Then there exist a function $\lambda(t) : [t_0, t_f] \to \mathbb{R}^n$, such that:

$$x'(t) = \frac{\partial H(x,u,\lambda)}{\partial \lambda}, x(t_0) = x_0 \quad (State\ \ Equations) \tag{3.4.3}$$

$$\lambda'(t) = -\frac{\partial H(x,u,\lambda)}{\partial x} \quad (Co - state\ \ Equations) \tag{3.4.4}$$

$$\lambda(t_f) = 0 \qquad (Transversality\ \ condition) \tag{3.4.5}$$

$$H(x^*, u^*, \lambda^*) = \max_{u \in U} H(x^*, u, \lambda^*) or \frac{\partial H}{\partial u} = 0 \tag{3.4.6}$$

Where $H(x,u,\lambda) = f(t,x,u) + \lambda(t)g(t,x,u)$ is called the Hamiltonian of the optimal control problem. Equation (3.4.6) is given in two forms because, when the Hamiltonian is differentiable with respect to u, the condition $\frac{\partial H}{\partial u} = 0$ can often be used to replace $H(x^*, u^*, \lambda^*) = \max_{u \in U} H(x^*, u, \lambda^*)$.

3.4.3 Optimal control in game addiction

The mathematical model of online game addiction has shown that the majority of addictions can be overcome through guidance and counseling for addicted individuals. If guidance and counseling are not given effectively and appropriately at the right time, overcoming OG addiction is difficult. The idea of optimum control has proven to be a useful instrument in understanding how to create the best possible intervention techniques that prevent the progression of OG addiction. With this strategy, the cost of OG addiction, the cost of putting control measures in place, or both, are minimized.

3.4.4 Numerical computation of optimal control problems

Since the solutions to many optimal control problems can not be obtained simply analytically, in order to solve optimal control problems in the form of (3.4.2) numerically, it involves finding piecewise continuous functions $u_i(t)$ that optimize the objective functional. Linear programming techniques are used for this purpose, bearing in mind that any solution to the problem must necessarily satisfy the state and co-state equations as well as the optimality conditions.

The optimality conditions can often be manipulated to find an explicit expression for the control variable $u(t)$, which can then be substituted into the state (3.4.3) and co-state systems (3.4.4) so that the two equations then form a two-point boundary value problem. Numerical methods for solving ordinary differential equations and boundary value problems can then be applied to solve the resulting two-point boundary value problem. The numerical scheme employed in this thesis to solve the resulting optimal control problems is the forward-backward sweep method, which is presented in [29].

CHAPTER 4
FORMULATION OF MATHEMATICAL MODEL

4.1 The existing mathematical model

The existing mathematical model of transmission dynamics of online game addiction using guidance and counseling proposed by Saman et al.[59], divided the population into four distinct epidemiological subclasses namely, susceptible (denoted by S) group of students who are prone to OG addiction, exposed (denoted by E) group of students who play online games, infected (denoted by I) group of students who are addicted to online games and recovered (denoted by R) group of students who overcome playing online games.

Susceptible students are recruited into the population at a rate of αN and exposed to playing online games via contact with infected (I) students at a rate of β. Exposed individuals with an addiction rate of γ become addicted and recover with a recovery rate of δ while the whole population has decreased in each compartment due to the same natural death rate of μ.

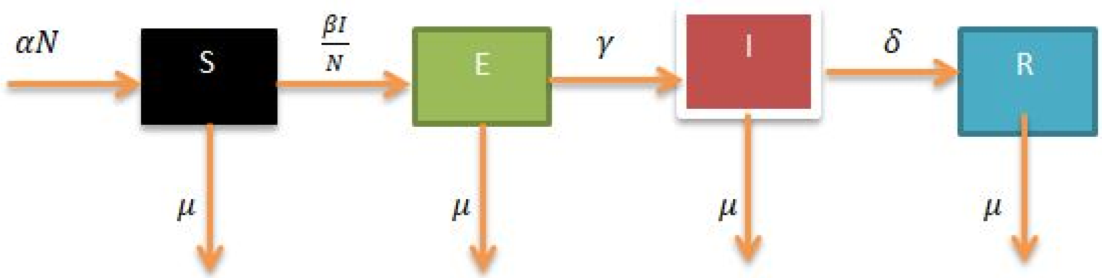


Figure 4.1: Flow chart of the existing mathematical model.

The spread of OG addiction is governed by the following system of non-linear ordinary differential equations:

$$\begin{aligned}\frac{dS}{dt} &= \alpha N - \frac{\beta IS}{N} - \mu S,\\ \frac{dE}{dt} &= \frac{\beta IS}{N} - \gamma E - \mu E,\\ \frac{dI}{dt} &= \gamma E + \delta I - \mu I,\\ \frac{dR}{dt} &= \delta I + \mu R,\end{aligned} \tag{4.1.1}$$

with initial conditions

$S(0) > 0$, $E(0) \geq 0$, $I(0) \geq 0$ and $R(0) \geq 0$.

4.2 The modified mathematical model

In this section, we modify the existing model to study the transmission dynamics of online game addiction in a population. Our model is an extension of the existing SEIR model for OG outbreaks. In the present model, we extended the SEIR model by including treatment, incompletely recovered and completely recovered classes assuming individuals commitment and effectiveness of treatment. We formulate and analyze the modified model. Further, we also perform a sensitivity analysis of the basic reproduction number for the model parameters.

4.2.1 Mathematical model formulation

We formulate a deterministic mathematical model for the transmission dynamics of OG addiction. We divided the total population into six sub-classes according to their gaming status, as follows: susceptible individuals (denoted by S) who have not yet been addicted to the addiction but are likely to get addicted in the future, exposed individuals (denoted by E) who play online games and are at risk of developing addiction, addicted individuals (denoted by A) are addicted to online games, treated individuals (denoted by T) are addicted individuals who join the treatment center, incompletely recovered individuals (denoted by Q_i) are incompletely recovered from OG addiction after taking the treatment and completely recovered individuals (denoted by Q_p) are completely recovered from OG addiction after taking the treatment.

The integration of the treated compartment, incomplete recovery compartment and complete recovery compartment in the SEIR model for online game addiction offers a more comprehensive insight into the dynamics of addiction and recovery within the online gaming environment. The treated compartment signifies individuals who have sought treatment for their addiction, enabling the analysis of various treatment strategies'(Cognitive-behavioral therapy, meditation, psychotherapy and Support groups) effectiveness in reducing addiction levels and facilitating recovery. The incomplete recovery compartment encompasses individuals who have made attempts to recover from their addiction but have not fully succeeded. Understanding the factors contributing to incomplete recovery can help identify barriers to successful recovery and inform the development of targeted interventions to support these individuals. The complete recovery compartment represents individuals who have successfully recovered from their addiction. Monitoring the transitions from the treatment compartment to the complete recovery compartment allows for an assessment of the overall effectiveness of interventions and policies aimed at reducing addiction rates and promoting long-term recovery.

Additionally, the following assumptions are crucial in the formulation of the model:

- The human population is divided into six compartments.
- The natural death rates are assumed to be the same for all the compartments.

- There is a constant inflow of people at risk of becoming susceptible(S) given by Λ.
- A susceptible individual starts playing the game when it comes into contact with an addicted and incompletely recovered individual with a different rate.
- Treated individuals temporarily recover from addiction due to a lack of commitment.
- Incompletely recovered individuals can be re-addicted due to escapism and dopamine release.
- Incompletely recovered individuals can completely recover from online game addiction because of a strong support system and clear boundaries.
- Treated individuals completely recovered from online game addiction due to the commitment they have and addressing underlying issues.
- Completely recovered individuals cannot become susceptible.
- The model's parameters are all non-negative.
- Sex and age structures are not considered.

Table 4.1: Modified model variables and their description

Variables	Description of the Variables
$S(t)$	The number of susceptible individuals at a time t
$E(t)$	The number of exposed individuals at a time t
$A(t)$	The number of addicted individual at a time t
$T(t)$	The number of treated individual at a time t
$Q_i(t)$	The number of incompletely recovered individual at a time t
$Q_p(t)$	The number of completely recovered individual at a time t

Table 4.2: Modified model parameters and their description

Parameters	Description of the Parameters
Λ	Recruitment rate
β_1	The rate at which individuals who are prone to online game addiction when in contact with online game addicted individuals start to play online games
β_2	The rate at which individuals who are prone to online game addiction when in contact with incompletely recovered individuals start to play online games
γ	The rate at which individuals who play online games become addicted
δ	The rate at which addicted individuals joined the treatment center
τ	Incomplete recovery rate of treated individuals
ω	Rate of re-addiction of the individuals from the incompletely recovered class
ε	Rate of complete recovery of the individuals from the incompletely recovered class
θ	Rate of complete recovery of treated individuals
μ	Natural mortality rate

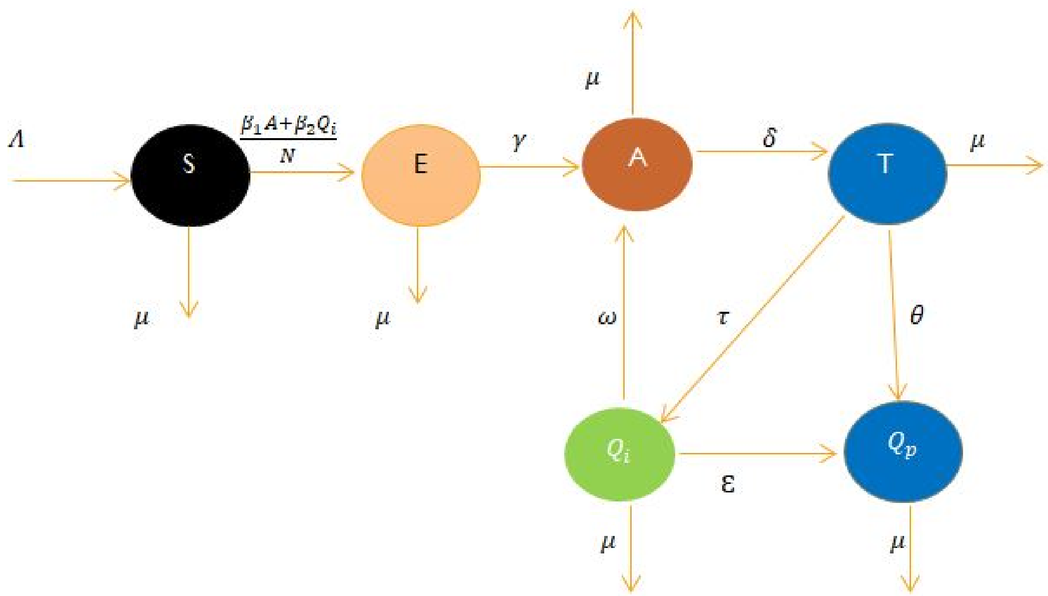


Figure 4.2: Flow chart of the modified mathematical model.

$$
\begin{aligned}
\frac{dS}{dt} &= \Lambda - \frac{(\beta_1 A + \beta_2 Q_i)S}{N} - \mu S, \\
\frac{dE}{dt} &= \frac{(\beta_1 A + \beta_2 Q_i)S}{N} - (\gamma + \mu)E, \\
\frac{dA}{dt} &= \gamma E + \omega Q_i - (\delta + \mu)A, \\
\frac{dT}{dt} &= \delta A - (\tau + \theta + \mu)T, \\
\frac{dQ_i}{dt} &= \tau T - (\omega + \varepsilon + \mu)Q_i, \\
\frac{dQ_p}{dt} &= \varepsilon Q_i + \theta T - \mu Q_p,
\end{aligned}
\tag{4.2.1}
$$

with initial conditions

$S(0) > 0$, $E(0) \geq 0$, $A(0) \geq 0$, $Q_i(0) \geq 0$ and $Q_p(0) \geq 0$.

4.3 Qualitative analysis of the model

In this section, we show the qualitative properties of the modified model. These analyses seek to show that the modified model is epidemiologically appropriate in the sense that the model and its predictions make sense. These analyses include the existence and uniqueness of the solution, boundedness of the solution, positivity of the solution, as well as the local and global stability of equilibrium points of the model (4.2.1).

4.3.1 Well-posedness

Theorem 4.3.1. **(The existence and uniqueness of solution)**
Let $\mathbb{D} = (S, E, A, T, Q_i, Q_p) \in \mathbb{R}^6_+$ denotes the region defined in (3.2.2) such that (3.2.3) and(3.2.4) hold. Then there exists a solution of model system (4.2.1) which is continuous and bounded in $\mathbb{D}$.

Proof. Suppose $\mathbb{D}$ denotes the above region. It is sufficient to show $\frac{\partial f_i}{\partial x_j}$ $i, j = 1, 2, ..., 6$ are continuous and bounded in $\mathbb{D}$. The region denoted contains

$$\mathbb{D} = (S, E, A, T, Q_i, Q_p).$$

Let

$$f_1 = \Lambda - \frac{(\beta_1 A + \beta_2 Q_i)S}{N} - \mu S, \tag{4.3.1}$$

$$f_2 = \frac{(\beta_1 A + \beta_2 Q_i)S}{N} - (\gamma + \mu)E, \tag{4.3.2}$$

$$f_3 = \gamma E + \omega Q_i - (\delta + \mu)A, \tag{4.3.3}$$

$$f_4 = \delta A - (\tau + \theta + \mu)T, \tag{4.3.4}$$

$$f_5 = \tau T - (\omega + \varepsilon + \mu)Q_i, \tag{4.3.5}$$

$$f_6 = \varepsilon Q_i + \theta T - \mu Q_p. \tag{4.3.6}$$

(4.3.7)

The system of equation (4.2.1) be represented by f_1, f_2, f_3, f_4, f_5 and f_6 respectively then, based on[18], from the first three equations in the system of equation (4.2.1) we obtain the following derivatives

$$\begin{aligned}
\frac{\partial f_1}{\partial S} &= -\frac{\beta_1 A + \beta_2 Q_i}{N} - \mu, \frac{\partial f_1}{\partial E} = 0, \frac{\partial f_1}{\partial A} = -\frac{\beta_1 S}{N}, \frac{\partial f_1}{\partial T} = 0, \\
\frac{\partial f_1}{\partial Q_i} &= -\frac{\beta_2 S}{N}, \frac{\partial f_1}{\partial Q_p} = 0, \frac{\partial f_2}{\partial S} = \frac{\beta_1 A + \beta_2 Q_i}{N}, \frac{\partial f_2}{\partial E} = -(\gamma + \mu), \\
\frac{\partial f_2}{\partial A} &= \frac{\beta_1 S}{N}, \frac{\partial f_2}{\partial T} = 0, \frac{\partial f_2}{\partial Q_i} = \frac{\beta_2 S}{N}, \frac{\partial f_2}{\partial Q_p} = 0 \frac{\partial f_3}{\partial S} = 0, \\
\frac{\partial f_3}{\partial E} &= \gamma, \frac{\partial f_3}{\partial A} = -(\delta + \mu), \frac{\partial f_3}{\partial T} = 0, \frac{\partial f_3}{\partial Q_i} = \omega, \frac{\partial f_2}{\partial Q_p} = 0.
\end{aligned}$$

$$\begin{aligned}
\left|\frac{\partial f_1}{\partial S}\right| &= \left|-\frac{\beta_1 A + \beta_2 Q_i}{N} - \mu\right| < \infty, \left|\frac{\partial f_1}{\partial E}\right| = |0| < \infty, \\
\left|\frac{\partial f_1}{\partial A}\right| &= \left|-\frac{\beta_1 S}{N}\right| < \infty, \left|\frac{\partial f_1}{\partial T}\right| = |0| < \infty, \left|\frac{\partial f_1}{\partial Q_i}\right| = \left|-\frac{\beta_2 S}{N}\right| < \infty, \\
\left|\frac{\partial f_1}{\partial Q_p}\right| &= |0| < \infty, \left|\frac{\partial f_2}{\partial S}\right| = \left|\frac{\beta_1 A + \beta_2 Q_i}{N}\right| < \infty, \\
\left|\frac{\partial f_2}{\partial E}\right| &= |-(\gamma + \mu)| < \infty, \left|\frac{\partial f_2}{\partial A}\right| = \left|\frac{\beta_1 S}{N}\right| < \infty, \left|\frac{\partial f_2}{\partial T}\right| = |0| < \infty, \\
\left|\frac{\partial f_2}{\partial Q_i}\right| &= \left|\frac{\beta_2 S}{N}\right| < \infty, \left|\frac{\partial f_2}{\partial Q_p}\right| = |0| < \infty \left|\frac{\partial f_3}{\partial S}\right| = |0| < \infty, \\
\left|\frac{\partial f_3}{\partial E}\right| &= |\gamma| < \infty, \left|\frac{\partial f_3}{\partial A}\right| = |-(\delta + \mu)| < \infty, \left|\frac{\partial f_3}{\partial T}\right| = |0| < \infty, \\
\left|\frac{\partial f_3}{\partial Q_i}\right| &= |\omega| < \infty, \left|\frac{\partial f_2}{\partial Q_p}\right| = |0| < \infty.
\end{aligned}$$

Similarly, we continue in the same manner for f_4, f_5 and f_6. The above partial derivatives are exist, continuous and bounded. Thus, the system of equations (4.2.1) are differentiable with respect to all state variables and all the partial derivatives of the system equations (4.2.1) are continuous and bounded. Therefor, the solution of system equation (4.2.1) exists and has a unique solution in region $\mathbb{D}$. □

Theorem 4.3.2. (**Positivity**) Let the initial conditions of the model system (4.2.1) are $S(0) > 0$, $E(0) \geq 0$, $A(0) \geq 0$, $T(0) \geq 0$, $Q_i(0) \geq 0$, $Q_p(0) \geq 0$, then the solution $S(t)$, $E(t)$, $A(t)$, $T(t)$, $Q_i(t)$, $Q_p(t)$ of the model system (4.2.1) are non-negative $\forall t \geq 0$.

Proof. We prove the non-negativity of the solution of the model (4.2.1) based on the approach of ([11],[12]). So, using this approach from the model system (4.2.1) of the first equation, we have

$$\frac{dS}{dt} = \Lambda - (\mu + \lambda)S. \tag{4.3.8}$$

where

$$\lambda = \frac{\beta_1 A + \beta_2 Q_i}{N}.$$

Re-arranging (4.3.8) yields

$$\frac{dS}{dt} + (\mu + \lambda)S = \Lambda. \tag{4.3.9}$$

Since (4.3.9) is a separable first-order differential equation for the state variable S. Finding the integrating factor yields $e^{\int(\mu+\lambda)dt}$, we obtain

$$S(t) = e^{-(\mu+\lambda)t}(\Lambda \int e^{(\mu+\lambda)t}dt). \tag{4.3.10}$$

Integrating and simplifying (4.3.10) we obtain

$$S(t) = e^{-(\mu+\lambda)t}K_1 + \frac{\Lambda}{\mu+\lambda}. \tag{4.3.11}$$

where K_1 is constant.

Computing (4.3.11) at $t = 0$ yields the following value of K_1

$$K_1 = S(0) - \frac{\Lambda}{\mu+\lambda} \tag{4.3.12}$$

Substituting (4.3.12) in to (4.3.11) we obtain

$$S(t) = S(0)e^{-(\mu+\lambda)t} + \frac{\Lambda}{\mu+\lambda}(1 - e^{-(\mu+\lambda)t}). \tag{4.3.13}$$

Since $S(0) > 0$, $e^{-(\mu+\lambda)t} > 0$ and $\frac{\Lambda}{\mu+\lambda} > 0 \; \forall t \geq 0$. Therefore,

$$S(t) = S(0)e^{-(\mu+\lambda)t} + \frac{\Lambda}{\mu+\lambda}(1 - e^{-(\mu+\lambda)t}) > 0.$$

From the model system (4.2.1) of the second equation, we have

$$\frac{dE}{dt} = \lambda S - (\gamma + \mu)E. \tag{4.3.14}$$

Re-arranging (4.3.14) yields

$$\frac{dE}{dt} + (\gamma + \mu)E = \lambda S. \tag{4.3.15}$$

Since (4.3.15) is a separable first-order differential equation for the state variable E. Finding the integrating factor yields $e^{\int(\gamma+\mu)dt}$, we obtain

$$E(t) = e^{-(\gamma+\mu)t}(\lambda S \int e^{(\gamma+\mu)t}dt). \tag{4.3.16}$$

Integrating and simplifying (4.3.16) we obtain

$$E(t) = e^{-(\gamma+\mu)t}K_2 + \frac{\lambda S}{\gamma+\mu}. \tag{4.3.17}$$

where K_2 is constant.

Computing (4.3.17) at $t = 0$ yields the following value of K_2

$$K_2 = E(0) - \frac{\lambda S}{\gamma + \mu} \tag{4.3.18}$$

Substituting (4.3.18) in to (4.3.17) we obtain

$$E(t) = E(0)e^{-(\gamma+\mu)t} + \frac{\lambda S}{\gamma + \mu}(1 - e^{-(\gamma+\mu)t}). \tag{4.3.19}$$

Since $E(0) \geq 0$, $e^{-(\gamma+\mu)t} > 0$ and $\frac{\lambda S}{\gamma+\mu} \geq 0 \; \forall t \geq 0$. Therefore,

$$E(t) = E(0)e^{-(\gamma+\mu)t} + \frac{\lambda S}{\gamma + \mu}(1 - e^{-(\gamma+\mu)t}) \geq 0.$$

From the model system (4.2.1) of the third equation, we have

$$\frac{dA}{dt} = \gamma E + \omega Q_i - (\delta + \mu)A. \tag{4.3.20}$$

Re-arranging (4.3.20) yields

$$\frac{dA}{dt} + (\delta + \mu)A = \gamma E + \omega Q_i. \tag{4.3.21}$$

Since (4.3.21) is a separable first-order differential equation for the state variable A. Finding the integrating factor yields $e^{\int(\delta+\mu)dt}$, we obtain

$$A(t) = e^{-(\delta+\mu)t}((\gamma E + \omega Q_i) \int e^{(\delta+\mu)t} dt). \tag{4.3.22}$$

Integrating and simplifying (4.3.22) we obtain

$$A(t) = e^{-(\delta+\mu)t} K_3 + \frac{\gamma E + \omega Q_i}{\delta + \mu}. \tag{4.3.23}$$

where K_3 is constant.

Computing (4.3.23) at $t = 0$ yields the following value of K_3

$$K_3 = A(0) - \frac{\gamma E + \omega Q_i}{\delta + \mu} \tag{4.3.24}$$

Substituting (4.3.24) in to (4.3.23) we obtain

$$A(t) = A(0)e^{-(\delta+\mu)t} + \frac{\gamma E + \omega Q_i}{\delta + \mu}(1 - e^{-(\delta+\mu)t}). \tag{4.3.25}$$

Since $A(0) \geq 0$, $e^{-(\delta+\mu)t} > 0$ and $\frac{\gamma E+\omega Q_i}{\delta+\mu} \geq 0 \; \forall t \geq 0$. Therefore,

$$A(t) = A(0)e^{-(\delta+\mu)t} + \frac{\gamma E + \omega Q_i}{\delta + \mu}(1 - e^{-(\delta+\mu)t}) \geq 0.$$

Following the same approach for $T(t), Q_i(t)$, and $Q_p(t)$, we obtain

$$\begin{aligned}
T(t) &= T(0)e^{-(\tau+\theta+\mu)t} + \frac{\delta A}{\tau + \theta + \mu}(1 - e^{-(\tau+\theta+\mu)t}) \geq 0. \\
Q_i(t) &= Q_i(0)e^{-(\omega+\varepsilon+\mu)t} + \frac{\tau T}{\omega + \varepsilon + \mu}(1 - e^{-(\omega+\varepsilon+\mu)t}) \geq 0. \\
Q_p(t) &= Q_p(0)e^{-\mu t} + \frac{\varepsilon Q_i + \theta T}{\mu}(1 - e^{-\mu t}) \geq 0.
\end{aligned}$$

Therefore, the solutions $S(t), E(t), A(t), T(t), Q_i(t), Q_p(t)$ in the system (4.2.1) remain non-negative for all $t \geq 0$. □

Theorem 4.3.3. (**Boundedness**) All the solutions $S(t)$, $E(t)$, $A(t)$, $T(t)$, $Q_i(t)$, $Q_p(t)$ of the model (4.2.1) are bounded.

Proof. In order to show that the population size of each compartment is bounded, we prefer to show that the total population size of the whole system, $N(t)$ is bounded. The total population is given by:

$$N(t) \;=\; S(t) + E(t) + A(t) + T(t) + Q_i(t) + Q_p(t). \tag{4.3.26}$$

Differentiating (4.3.26) both sides with respect to time t and inserting (4.2.1) and after some simplification we obtain

$$\frac{dN(t)}{dt} = \Lambda - \mu N(t). \tag{4.3.27}$$

Re-arranging (4.3.27) yields

$$\frac{dN(t)}{dt} + \mu N(t) = \Lambda. \tag{4.3.28}$$

Since (4.3.28) is a linear first-order differential equation, finding the integrating factor, yields $e^{\mu t}$. Multiplying both sides of the equation (4.3.28) by $e^{\mu t}$ and integrating both sides, we obtain

$$N(t) \;\leq\; \frac{\Lambda}{\mu} + N(0)e^{-\mu t}. \tag{4.3.29}$$

As $t \to \infty$, then $N(t) \to \frac{\Lambda}{\mu}$(4.3.29). Which implies, $0 \leq N(t) \leq \frac{\Lambda}{\mu}$.

Thus, the feasible solution set of the model system (4.2.1) remain in the region Ω for all time t, $\Omega = \{(S, E, A, T, Q_, Q_p) \in \mathbb{R}^6_+; 0 \leq N(t) \leq \frac{\Lambda}{\mu}\}$.

Therefore, every solution with condition to $\mathbb{R}^6_+$, the region is positively invariant. It is sufficient to study the dynamics for online game addiction model in $\mathbb{R}^6_+$. □

4.3.2 Steady state

The steady states of the system (4.2.1) are solutions of the following equations:

$$\begin{aligned}
\Lambda - \frac{(\beta_1 A + \beta_2 Q_i)S}{N} - \mu S &= 0\\
\frac{(\beta_1 A + \beta_2 Q_i)S}{N} - (\gamma + \mu)E &= 0\\
\gamma E + \omega Q_i - (\delta + \mu)A &= 0\\
\delta A - (\tau + \theta + \mu)T &= 0\\
\tau T - (\omega + \varepsilon + \mu)Q_i &= 0\\
\varepsilon Q_i + \theta T - \mu Q_p &= 0.
\end{aligned}$$

There are at most two steady states for the system (4.2.1): the OG addiction free equilibrium (E_0) and endemic equilibrium (E_1).

4.3.3 OG addiction free equilibrium point

The OG addiction-free equilibrium point of our model is obtained by setting the online gamer state variables $E^0 = 0$, $A^0 = 0$, $T^0 = 0$ and $Q_i^0 = 0$. Thus, the OG addiction-free equilibrium point of the model (4.2.1) is given by

$$E_0 = (S^0, E^0, A^0, T^0, Q_i^0, Q_p^0) = (\frac{\Lambda}{\mu}, 0, 0, 0, 0, 0).$$

4.3.4 Basic reproduction number

The basic reproduction number, denoted by R_0, defined as the average number of secondary addictions produced when a single addicted individual is introduced into a susceptible population. We compute R_0 using the next-generation matrix method in Diekmann[9]. The next generation matrix comprises two matrices, F and V, whose elements in the matrix constitute the new players that will arise and the transfer of players from one compartment to another, respectively.

Let us define f_i and v_i as follows:

$i)$ f_i be the rate of appearance of new players in compartment i.

$ii)$ v_i^+ be the rate transfer of individuals into compartment i.

$iii)$ v_i^- be the rate transfer of individuals out of compartment i.

$iv)$ v_i be the transfer of individuals in and out of compartment i.
i.e., $v_i = v_i^- - v_i^+$

Now, based on ([13],[33],[59]) from the model system (4.2.1), online game playing compartments are:

$$\begin{aligned} E' &= \frac{(\beta_1 A + \beta_2 Q_i)S}{N} - (\gamma + \mu)E, \\ A' &= \gamma E + \omega Q_i - (\delta + \mu)A, \\ T' &= \delta A - (\tau + \theta + \mu)T, \\ Q_i' &= \tau T - (\omega + \varepsilon + \mu)Q_i \end{aligned}$$

$$f_i = \begin{bmatrix} \frac{(\beta_1 A + \beta_2 Q_i)S}{N} \\ 0 \\ 0 \\ 0 \end{bmatrix}, F = \frac{\partial f_i(E_0)}{\partial E, A, T, Q_i} = \begin{bmatrix} 0 & \beta_1 & 0 & \beta_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$v_i = \begin{bmatrix} (\gamma + \mu)E \\ -\gamma E - \omega Q_i + (\delta + \mu)A \\ -\delta A + (\tau + \theta + \mu)Q_i \\ -\tau T + (\omega + \epsilon + \mu)Q_i \end{bmatrix}, V = \frac{\partial v_i(E_0)}{\partial E, A, T, Q_i} = \begin{bmatrix} k_1 & 0 & 0 & 0 \\ -\gamma & k_2 & 0 & -\omega \\ 0 & -\delta & k_3 & 0 \\ 0 & 0 & -\tau & k_4 \end{bmatrix},$$

$$V^{-1} = \begin{bmatrix} \frac{1}{k_1} & 0 & 0 & 0 \\ a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix}.$$

where

$$\begin{aligned}
k_1 &= \gamma+\mu, k_2 = \delta+\mu, k_3 = \tau+\theta+\mu, k_4 = \omega+\epsilon+\mu, \\
a_1 &= \frac{\gamma k_3 k_4}{|V|}, a_2 = \frac{k_1 k_3 k_4}{|V|}, a_3 = \frac{k_1 \tau \omega}{|V|}, a_4 = \frac{\delta \omega k_2}{|V|}, a_5 = -\frac{\gamma \delta k_4}{|V|}, \\
a_6 &= \frac{\delta k_1 k_4}{|V|}, a_7 = \frac{k_1 k_2 k_4}{|V|}, a_8 = \frac{\delta \omega k_1}{|V|}, a_9 = \frac{\gamma \delta \tau}{|V|}, a_{10} = \frac{\delta \tau k_1}{|V|}, \\
a_{11} &= \frac{\delta k_2 k_2}{|V|}, a_{12}\frac{k_1 k_2 k_3}{|V|}, |V| = k_1(k_2 k_3 k_4 - \omega \delta \tau).
\end{aligned}$$

Therefore, the next generation matrix is given by

$$FV^{-1} = \begin{bmatrix} a_1\beta_1 + a_9\beta_2 & a_2\beta_1 + a_{10}\beta_2 & a_3\beta_1 + a_{11}\beta_2 & a_4\beta_1 + a_{12}\beta_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We find the eigenvalues of FV^{-1} by solving the characteristics equation $|FV^{-1} - \lambda I_4| = 0$, where I_4 is a 4 by 4 identity matrix, as $\lambda_1 = a_1\beta_1 + a_9\beta_2$ and $\lambda_{2,3,4,5,6} = 0$. The basic reproduction number is the spectra radius of FV^{-1} given by

$$R_0 = \gamma \frac{k_3 k_4 \beta_1 + \delta \tau \beta_2}{k_1(k_2 k_3 k_4 - \omega \delta \tau)}.$$

Therefore, we can rewrite the the basic reproduction number as follow

$$R_0 = \gamma \frac{(\tau+\theta+\mu)(\omega+\epsilon+\mu)\beta_1 + \delta\tau\beta_2}{(\gamma+\mu)((\delta+\mu)(\tau+\theta+\mu)(\omega+\epsilon+\mu) - \omega\delta\tau)}. \tag{4.3.30}$$

4.3.5 Local stability of OG addiction-free equilibrium

Theorem 4.3.4. The OG addiction free equilibrium point E_0 of the system (4.2.1) is locally asymptotically stable if $R_0 < 1$.

Proof. The Jacobian matrix of the system (4.2.1) at OG addiction- free equilibrium point E_0 is given by

$$J(E_0) = \begin{bmatrix} -\mu & 0 & -\beta_1 & 0 & -\beta_2 & 0 \\ 0 & -k_1 & \beta_1 & 0 & \beta_2 & 0 \\ 0 & \gamma & -k_2 & 0 & \omega & 0 \\ 0 & 0 & \delta & -k_3 & 0 & 0 \\ 0 & 0 & 0 & \tau & -k_4 & 0 \\ 0 & 0 & 0 & \theta & \epsilon & -\mu \end{bmatrix}.$$

Clearly, the two eigenvalues $\lambda_{1,2} = -\mu$ and the remaining eigenvalues $\lambda_3, \lambda_4, \lambda_5, \lambda_6$ can be obtained from the following reduced matrix characteristics equation

$$J_4(E_0) = \begin{bmatrix} -k_1 & \beta_1 & 0 & \beta_2 \\ \gamma & -k_2 & 0 & \omega \\ 0 & \delta & -k_3 & 0 \\ 0 & 0 & \tau & -k_4 \end{bmatrix}.$$

Thus, the characteristics equation of $J_4(E_0)$ is given by $|J_4(E_0) - \lambda I_4|$=0, where I_4 is a 4 by 4 identity matrix.

$$p(\lambda) = c_4\lambda^4 + c_3\lambda^3 + c_2\lambda^2 + c_1\lambda + c_0,$$

where

$$\begin{aligned} c_4 &= 1, \\ c_3 &= k_1 + k_2 + k_3 + k_4, \\ c_2 &= k_3k_4 + k_2k_3 + k_2k_4 + k_1(k_2 + k_3 + k_4) - \beta_1\gamma, \\ c_1 &= k_1(k_3k_4 + k_2k_3 + k_2k_4) - k_2k_3k_4 - \beta_1\gamma(k_3 + k_4), \\ c_0 &= k_1k_2k_3k_4 - k_1\omega\delta\tau - \beta_1\gamma k_4k_3 - \beta_2\gamma\delta\tau. \end{aligned}$$

Using Routh-Hurwitz criterion, it can be observed that all the eigenvalues of the characteristic equation have a negative real part if $c_3 > 0, c_1 > 0, c_0 > 0$ and $c_1c_2c_3 > c_1^2 + c_3^2c_0$. We can see that c_3 is positive because all parameters are positive and their sum is also positive. But in order to say c_1 and c_0 positive $k_1(k_3k_4 + k_2k_3 + k_2k_4) - k_2k_3k_4 - \beta_1(k_3 + k_4)$ and $k_1k_2k_3k_4 - k_1\omega\delta\tau - \beta_1\gamma k_4k_3 - \beta_2\gamma\delta\tau$ must be positive, respectively. Hence, $c_1 > 0$ if it fulfills that $k_1(k_3k_4 + k_2k_3 + k_2k_4) > k_2k_3k_4 + \beta_1(k_3 + k_4)$ and $c_0 > 0$ if it fulfills that $k_1k_2k_3k_4 - k_1\omega\delta\tau > \beta_1\gamma k_4k_3 + \beta_2\gamma\delta\tau$. Additionally, $c_1c_2c_3 > c_1^2 + c_3^2c_0$. From the Routh-Hurwitz criterion, it follows that $c_1^2 + c_3^2c_0 > 0$ if $c_0 = k_1k_2k_3k_4 - k_1\omega\delta\tau - \beta_1\gamma k_4k_3 - \beta_2\gamma\delta\tau > 0$. From this we get $1 > \gamma\frac{\beta_1k_4k_3 + \beta_2\delta\tau}{k_1(k_2k_3k_4 - \omega\delta\tau)}$ or $(1 > \gamma\frac{(\tau+\theta+\mu)(\omega+\epsilon+\mu)\beta_1 + \delta\tau\beta_2}{(\gamma+\mu)((\delta+\mu)(\tau+\theta+\mu)(\omega+\epsilon+\mu) - \omega\delta\tau)})$ which means $1 > R_0$. This implies that $c_1^2 + c_3^2c_0 > 0$ if $c_0 > 0$ for $R_0 < 1$.

Therefore, all the eigenvalues of the characteristic equation are clearly real and negative if $R_0 < 1$. We can conclude that the OG addiction equilibrium point of the system (4.2.1) is locally asymptotically stable if $R_0 < 1$. □

4.3.6 Global stability of OG addiction-free equilibrium

Theorem 4.3.5. For $R_0 < 1$, the OG addiction-free equilibrium E_0 of the system (4.2.1) is globally asymptotically stable if $S^0 \geq S$.

Proof. Let us rewrite the system has

$$\begin{aligned} \frac{dZ_1}{dt} &= F(Z_1, Z_2), \\ \frac{dZ_2}{dt} &= G(Z_1, Z_2), \quad G(Z_1, 0) = 0. \end{aligned}$$

where $Z_1 = (S, Q_p) \in \mathbb{R}^2_+$ is the class of non-gamer individuals, and $Z_2 = (E, A, T, Q_i) \in \mathbb{R}^4_+$ is the class of gamer individuals. The OG addiction-free equilibrium point of the model is denoted by $U_0 = (Z_1^*, 0)$, where $Z_1^* = (\frac{\Lambda}{\mu}, 0)$. In theorem (4.3.4), we proved that E_0 is locally asymptotically stable.

To prove the global stability of E_0, we apply Castillo-Chavez theorem stated in theorem(3.2.7) following the approach of Abayneh and Zerihun[13].

From the system (4.2.1) we have

$$\frac{dZ_1}{dt} = F(Z_1, Z_2) = \begin{bmatrix} \Lambda - \frac{(\beta_1 A + \beta_2 Q_i)S}{N} - \mu S \\ \varepsilon Q_i + \theta T - \mu Q_p \end{bmatrix},$$

$$\frac{dZ_1}{dt} = G(Z_1, Z_2) = \begin{bmatrix} \frac{(\beta_1 A + \beta_2 Q_i)S}{N} - (\gamma + \mu)E \\ \gamma E + \omega Q_i - (\delta + \mu)A \\ \delta A - (\tau + \theta + \mu)T \\ \tau T - (\omega + \varepsilon + \mu)Q_i \end{bmatrix}.$$

I. To show Z_1^* is globally asymptotically stable for the system $\frac{dZ_1}{dt} = F(Z_1, 0)$ we plug in $E = 0, A = 0, T = 0$ and $Q_i = 0$, then we obtain the reduced system

$$\frac{dZ_1}{dt} = F(Z_1, 0) = \begin{bmatrix} \Lambda - \mu S \\ -\mu Q_i \end{bmatrix}.$$

We obtain new system

$$\frac{dS(t)}{dt} = \Lambda - \mu S(t), \tag{4.3.31a}$$

$$\frac{dQ_p(t)}{dt} = -\mu Q_p(t). \tag{4.3.31b}$$

The system (4.3.31) is non-homogeneous linear system of ordinary differential equations. Applying the theorem (3.2.3) for the system (4.3.31) we obtain the following solutions

$$S(t) = \frac{\Lambda}{\mu} + (S(0) - \frac{\Lambda}{\mu})e^{-\mu t},$$

$$Q_p(t) = Q_p(0)e^{-\mu t}.$$

Taking as $t \to \infty$, we obtain $\lim_{t\to\infty}(S(t), Q_p(t)) = (\frac{\Lambda}{\mu}, 0) = Z_1^*$.
Therefore, Z_1^* is globally asymptotically stable to the system $\frac{dZ_1}{dt} = F(Z_1, 0)$.

II. We will show $G(Z_1, Z_2) = MZ_2 - \hat{G}(Z_1, Z_2)$, $\hat{G}(Z_1, Z_2) \geq 0$ for $(Z_1, Z_2) \in \Omega$. Where $M = \frac{\partial\, G(Z_1^*, 0)}{\partial Z_2}$ is a Metzler matrix whose off-diagonal elements are non-negative and Ω is the region where the model makes biological sense.

Consider a matrix

$$M = \frac{\partial G(Z_1^*, 0)}{\partial Z_2} = \begin{bmatrix} -(\gamma+\mu) & \beta_1 & 0 & \beta_2 \\ \gamma & -(\delta+\mu) & 0 & \omega \\ 0 & \delta & -(\tau+\theta+\mu) & 0 \\ 0 & 0 & \tau & -(\omega+\epsilon+\mu) \end{bmatrix}.$$

Hence, M is a Metzer matrix (off diagonal elements are non-negative). Here,

$$\hat{G}(Z_1, Z_2) = MZ_2 - G(Z_1, Z_2).$$

After performing some simplifications, we obtain

$$\hat{G}(Z_1, Z_2) = \begin{bmatrix} (\beta_1 A + \beta_2 Q_i)\frac{S^0}{N} - (\beta_1 A + \beta_2 Q_i)\frac{S}{N} \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

This implies that

$$\hat{G}(Z_1, Z_2) = (S^0 - S) \begin{bmatrix} \frac{\beta_1 A + \beta_2 Q_i}{N} \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Since $S \leq S^0$, we have $(S^0 - S)\frac{\beta_1 A + \beta_2 Q_i}{N} \geq 0$; the preceding matrix is non-negative. Therefore, it can be written as

$$\hat{G}(Z_1, Z_2) = (S^0 - S) \begin{bmatrix} \frac{\beta_1 A + \beta_2 Q_i}{N} \\ 0 \\ 0 \\ 0 \end{bmatrix} \geq 0.$$

Therefore, the OG addiction-free equilibrium point E^0 is globally asymptotically stable for $R_0 < 1$ by the Castillo-Chavez theorem (3.2.7).

□

4.3.7 Endemic equilibrium point

Endemic equilibrium point is a steady state solution where the OG addiction persists in the population. In the presence of OG addiction in the population, there exist an equilibrium point called endemic equilibrium point denoted by $E_1 = (S^*, E^*, A^*, T^*, Q_i^*, Q_p^*)$. Based on approach of Abayneh and Zerihun[13], it can be obtained by setting each equation of the system (4.2.1) equal to zero:

$$\Lambda - \frac{(\beta_1 A + \beta_2 Q_i)S}{N} - \mu S = 0 \tag{4.3.32a}$$

$$\frac{(\beta_1 A + \beta_2 Q_i)S}{N} - (\gamma + \mu)E = 0 \tag{4.3.32b}$$

$$\gamma E + \omega Q_i - (\delta + \mu)A = 0 \tag{4.3.32c}$$

$$\delta A - (\tau + \theta + \mu)T = 0 \tag{4.3.32d}$$

$$\tau T - (\omega + \varepsilon + \mu)Q_i = 0 \tag{4.3.32e}$$

$$\varepsilon Q_i + \theta T - \mu Q_p = 0 \quad . \tag{4.3.32f}$$

From equation (4.3.32d), we have

$$A^* = \frac{k_3 T^*}{\delta}. \tag{4.3.33}$$

From equation (4.3.32e), we have

$$Q_i^* = \frac{\tau T^*}{k_4}. \tag{4.3.34}$$

By inserting equation (4.3.34) into the equation (4.3.32f), we obtain

$$Q_p^* = \frac{\epsilon\tau + \theta k_4}{\mu k_4} T^*. \tag{4.3.35}$$

By substituting equations (4.3.33) and (4.3.34) into the equation (4.3.32c), we obtain

$$E^* = \frac{k_2 k_3 k_4 - \omega\tau\delta}{\gamma\delta k_4} T^*. \tag{4.3.36}$$

By substituting equations (4.3.33), (4.3.34), and (4.3.36) into the equation (4.3.32b) and doing some simplification, we obtain

$$S^* = \frac{\Lambda}{\mu R_0}. \tag{4.3.37}$$

By inserting equations(4.3.33), (4.3.34), and (4.3.37) into the equation (4.3.32a) and doing some simplification, we obtain

$$T^* = \frac{\Lambda\delta k_4}{\beta_1 k_3 k_4 + \beta_2 \tau\delta}(1 - \frac{1}{R_0}). \tag{4.3.38}$$

Therefore, the endemic equilibrium point is given by

$$E_1 = (S^*, E^*, A^*, T^*, Q_i^*, Q_p^*).$$

where

$$\begin{aligned}
S^* &= \frac{\Lambda}{\mu R_0}, \\
E^* &= \frac{k_2k_3k_4 - \omega\tau\delta}{\gamma\delta k_4}T^*, \\
A^* &= \frac{k_3T^*}{\delta}, \\
T^* &= \frac{\Lambda\delta k_4}{\beta_1k_3k_4 + \beta_2\tau\delta}(1 - \frac{1}{R_0}), \\
Q_i^* &= \frac{\tau T^*}{k_4}, \\
Q_p^* &= \frac{\epsilon\tau + \theta k_4}{\mu k_4}T^*.
\end{aligned}$$

From this, we see that the endemic equilibrium exists if $R_0 > 1$. And the force of addiction can be written as

$$\lambda^* = \frac{(\beta_1A^* + \beta_2Q_i^*)}{N}. \tag{4.3.39}$$

Substituting A^* and Q_i^* into (4.3.39) yields

$$\lambda^* = \mu(1 - \frac{1}{R_0}), \tag{4.3.40}$$

provided that $R_0 > 1$.

4.3.8 Local stability of endemic equilibrium

Theorem 4.3.6. The endemic equilibrium point of model (4.2.1) is locally asymptotically stable if $R_0 > 1$ and unstable if $R_0 < 1$.

Proof. To study local stability at the endemic equilibrium point(E_1) of the dynamical system, we evaluate the Jacobian matrix of the system (4.2.1) at the endemic equilibrium point E_1 following the approach in ([1],[13]) and it is given by

$$J(E_1) = \begin{bmatrix} -s-\mu & 0 & -q & 0 & -r & 0 \\ s & -k_1 & q & 0 & r & 0 \\ 0 & \gamma & -k_2 & 0 & \omega & 0 \\ 0 & 0 & \delta & -k_3 & 0 & 0 \\ 0 & 0 & 0 & \tau & -k_4 & 0 \\ 0 & 0 & 0 & \theta & \epsilon & -\mu \end{bmatrix}.$$

where

$$\begin{aligned}
q &= \frac{\beta_1S^*}{N}, \\
r &= \frac{\beta_2Q_i^*}{N}, \\
s &= \frac{\beta_1A^* + \beta_2Q_i^*}{N}.
\end{aligned}$$

We obtain the characteristics equation of $J(E_1)$ from $|J(E_1) - \lambda I_6| = 0$, where I_6 is a 6 by 6 identity matrix.

$$p(\lambda) = m_6\lambda^6 + m_5\lambda^5 + m_4\lambda^4 + m_3\lambda^3 + m_2\lambda^2 + m_1\lambda + m_0,$$

where

$$\begin{aligned}
m_6 &= 1,\\
m_5 &= k_1 + k_2 + k_3 + k_4 + 2\mu + s,\\
m_4 &= \mu^2 + k_1(k_2 + k_4 + 2\mu) + k_2(k_3 + k_4 + 2\mu) + 2\mu(k_3 + k_4) - \gamma q + s(k_1\\
&\quad + k_2 + k_3 + k_4 + \mu),\\
m_3 &= \mu^2(k_1 + k_2 + k_3 + k_4 + k_1k_2k_3 + k_1k_2k_4 + k_2k_3k_4 + \mu(2k_1k_2 + 2k_1k_3)\\
&\quad + 2k_1k_4 + 2k_2k_3 + 2k_2k_4 + 2k_3k_4 - 2\gamma) - \delta\omega\tau - \gamma k_3 q + s(k_1k_2 + k_1k_3\\
&\quad + k_1k_4 + k_2k_3 + k_2k_4 + k_3k_4 + k_1\mu + k_2\mu + k_3\mu + k_4\mu),\\
m_2 &= \mu^2(k_1k_2 + k_2k_3 + k_1k_4 + k_2k_3 + k_2k_4 - \gamma q) + k_1k_2k_3k_4 - \gamma k_3k_4q +\\
&\quad \mu(k_1k_2k_3 + 2k_1k_2k_4 + 2k_2k_3k_4 - 2\delta\omega\tau) + s(\mu(k_1k_2 + k_1k_3 + k_1k_4 + k_2k_3\\
&\quad + k_2k_4 + k_3k_4) - \delta\omega\tau)\mu + k_2\mu + k_3\mu + k_4\mu),\\
m_1 &= \mu^2(k_1k_2k_3 + k_1k_3k_4 + k_2k_3k_4 - \delta\omega\tau - \gamma k_3q - \gamma k_4q) + \mu(k_1k_2k_3s +\\
&\quad k_1k_2k_4s + k_1k_3k_4s + k_2k_3k_4s - \delta\omega\tau s) - \delta k_1\omega\tau s,\\
m_0 &= k_1k_2k_3k_4\mu^2 - \delta k_1\mu\omega\tau s + k_1k_2k_4\mu^2 - \delta k_1\mu^2\omega\tau - \gamma k_3k_4\mu^2 q - \delta\gamma\tau r.
\end{aligned}$$

Now for the characteristics polynomial equation

$$p(\lambda) = m_6\lambda^6 + m_5\lambda^5 + m_4\lambda^4 + m_3\lambda^3 + m_2\lambda^2 + m_1\lambda + m_0,$$

the alternative Routh–Hurwitz array is as follow

λ^6	1	m_4	m_2	m_0
λ^5	m_5	m_3	m_1	0
λ^4	b_1	b_2	b_3	0
λ^3	c_1	c_2	0	
λ^2	d_1	d_2		
λ^1	e_1	0		
λ^0	m_0			

where

$$\begin{aligned}
b_1 &= \frac{m_5m_4 - m_3 \times 1}{m_5}, & b_2 &= \frac{m_5m_2 - m_1 \times 1}{m_5},\\
b_3 &= \frac{m_5m_0 - 0 \times 1}{m_5}, & c_1 &= \frac{b_1m_3 - b_2m_5}{b_1},\\
c_2 &= \frac{b_1m_1 - m_5b_3}{b_1}, & d_1 &= \frac{c_1b_2 - b_1c_2}{c_1},\\
d_2 &= \frac{c_1b_3 - 0 \times b_2}{c_1}, & e_1 &= \frac{c_2d_1 - c_1d_2}{d_1}.
\end{aligned}$$

When all the first column elements of the Routh-Hurwitz array have the same algebraic signs, then we can conclude that the endemic equilibrium point is stable, which means that OG addiction exists in the population. This condition satisfied if $b_1 > 0$, $c_1 > 0$, $d_1 > 0$, $e_1 > 0$ and $m_0 > 0$ because $m_5 = k_1 + k_2 + k_3 + k_4 + 2\mu + s > 0$. Additionally, $m_0 > 0$, $m_3 > 0$ and $m_4 > 0$ when $k_1k_2k_3k_4\mu^2 + k_1k_2k_4\mu^2 > \delta k_1\mu\omega\tau s + \delta k_1\mu^2\omega\tau + \gamma k_3k_4\mu^2 q + \delta\gamma\tau r$, $\mu^2(k_1+k_2+k_3+k_4+k_1k_2k_3+k_1k_2k_4+k_2k_3k_4+\mu(2k_1k_2+2k_1k_3)+2k_1k_4+2k_2k_3+2k_2k_4+2k_3k_4-2\gamma)+s(k_1k_2+k_1k_3+k_1k_4+k_2k_3+k_2k_4+k_3k_4+k_1\mu+k_2\mu+k_3\mu+k_4\mu) > \delta\omega\tau+\gamma k_3 q$ and $\mu^2+k_1(k_2+k_4+2\mu)+k_2(k_3+k_4+2\mu)+2\mu(k_3+k_4)+s(k_1+k_2+k_3+k_4+\mu) > \gamma q$, respectively.

□

Further, in the bifurcation analysis section(4.3.10), stability is discussed and simulation shows that E_1 is locally asymptotically stable when $R_0 > 1$.

4.3.9 Global stability of endemic equilibrium

Theorem 4.3.7. If $R_0 > 1$, then the endemic equilibrium point of the system is globally asymptotically stable on Ω.

Proof. To show the endemic equilibrium point (E_1) globally asymptotically stable we construct the Lyapunov function as following:

$$\begin{aligned} L(S^*, E^*, A^*, T^*, Q_i^*, Q_p^*) =& S - S^* + S^* \times ln(\frac{S^*}{S}) \\ &+ E - E^* + E^* \times ln(\frac{E^*}{E}) \\ &+ A - A^* + A^* \times ln(\frac{A^*}{A}) \\ &+ T - T^* + T^* \times ln(\frac{T^*}{T}) \\ &+ Q_i - Q_i^* + Q_i^* \times ln(\frac{Q_i^*}{Q_i}) \\ &+ Q_p - Q_p^* + Q_p^* \times ln(\frac{Q_p^*}{Q_p}). \end{aligned} \tag{4.3.41}$$

Differentiating (4.3.41) we obtain:

$$\begin{aligned} \frac{dL}{dt} =& \frac{S - S^*}{S}\frac{dS}{dt} + \frac{E - E^*}{E}\frac{dE}{dt} + \frac{A - A^*}{A}\frac{dA}{dt} \\ &+ \frac{T - T^*}{T}\frac{dT}{dt} + \frac{Q_i - Q_i^*}{Q_i}\frac{dQ_i}{dt} + \frac{Q_p - Q_p^*}{Q_p}\frac{dQ_p}{dt}. \end{aligned} \tag{4.3.42}$$

By inserting (4.2.1) and simplifying (4.3.42) we get

$$\frac{dL}{dt} = P - M, \tag{4.3.43}$$

where P and M be positive and negative terms of (4.3.43) respectively. That is

$$\begin{aligned}P = \Lambda &+ \frac{\beta_1 A + \beta_2 Q_i}{N}(S + S^*) + \mu S^* + (\gamma+\mu)E^* + \gamma E + \omega Q_i \\ &+ (\delta+\mu)A^* + \delta A + (\tau+\theta+\mu)T^* + \tau T + (\omega+\varepsilon+\mu)Q_i^* \\ &+ \varepsilon Q_i + \theta T + \mu Q_p^*,\end{aligned}$$

$$\begin{aligned}M = &\frac{(\beta_1 A + \beta_2 Q_i)S}{N}(1 + \frac{E^*}{E}) + \mu S + \Lambda\frac{S^*}{S} + (\gamma+\mu)E + (\delta+\mu)A \\ &+ \gamma E\frac{A^*}{A} + \omega Q_i\frac{A^*}{A} + (\tau+\theta+\mu)T + \delta A\frac{T^*}{T} + \mu Q_p + \varepsilon Q_i\frac{Q_p^*}{Q_p^*} + \\ &\theta T\frac{Q_p^*}{Q_p} + (\omega+\varepsilon+\mu)Q_i + \tau T\frac{Q_p^*}{Q_p}.\end{aligned}$$

If $P < M$, then $\frac{dL}{dt} < 0$ and $\frac{dL}{dt}$ =0 if and only if $S = S^*$, $E = E^*$, $A = A^*$, $T = T^*$, $Q_i = {Q_i}^*$ and $Q_p = {Q_p}^*$. The largest compact invariant set in $(S^*, E^*, A^*, T^*, Q_i^*, Q_P^*) \in {\mathbb{R}^6}_+ : \frac{dL}{dt} = 0$ is the singleton of E_1. This implies, based on Sastry[60], that E_1 is globally asymptotically stable in ${\mathbb{R}^6}_+$ if $P < M$ by LaSalle's invariant principle. □

4.3.10 Bifurcation analysis

Based on([7],[64]), we used central manifold theory(theorem (3.3.1)) to verify the backward and forward bifurcation. We perform bifurcation analysis by taking β_1 as a bifurcation parameter and implementing the following change of variables on the system (4.2.1).

Let $S = x_1, E = x_2, A = x_3, T = x_4, Q_i = x_5$ and $Q_p = x_6$. Moreover, by using vector notation $x = (x_1, x_2, x_3, x_4, x_5, x_6)^T$, the system (4.2.1) can be written in the form $\frac{dx}{dt} = F(x)$, with $F = (f_1, f_2, f_3, f_4, f_5, f_6)^T$, as shown below:

$$\begin{aligned}\frac{dx_1}{dt} &= \Lambda - \frac{(\beta_1 x_3 + \beta_2 x_5)x_1}{N} - \mu x_1, \\ \frac{dx_2}{dt} &= \frac{(\beta_1 x_3 + \beta_2 x_5)x_1}{N} - (\gamma+\mu)x_2, \\ \frac{dx_3}{dt} &= \gamma x_2 + \omega x_5 - (\delta+\mu)x_3, \\ \frac{dx_4}{dt} &= \delta x_3 - (\tau+\theta+\mu)x_4, \\ \frac{dx_5}{dt} &= \tau x_4 - (\omega+\varepsilon+\mu)x_5, \\ \frac{dx_6}{dt} &= \varepsilon x_5 + \theta x_4 - \mu x_6.\end{aligned} \tag{4.3.44}$$

We consider $\beta_1 = \beta^*$ as bifurcation parameter. Solving for β^* from $R_0 = 1$ we obtain

$$\beta^* = \frac{(\gamma+\mu)((\delta+\mu)(\tau+\theta+\mu)(\omega+\epsilon+\mu) - \omega\delta\tau) - \gamma\delta\tau\beta_2}{\gamma(\tau+\theta+\mu)(\omega+\epsilon+\mu)}.$$

The Jacobian matrix of the system (4.3.44) at OG addiction-free equilibrium point E_0 with

$\beta_1 = \beta^*$ is give by

$$J^*(E_0) = \begin{bmatrix} -\mu & 0 & -\beta^* & 0 & -\beta_2 & 0 \\ 0 & -k_1 & \beta^* & 0 & \beta_2 & 0 \\ 0 & \gamma & -k_2 & 0 & \omega & 0 \\ 0 & 0 & \delta & -k_3 & 0 & 0 \\ 0 & 0 & 0 & \tau & -k_4 & 0 \\ 0 & 0 & 0 & \theta & \epsilon & -\mu \end{bmatrix}.$$

The Jacobian matrix $J^*(E_0)$ of the linearized system has a simple zero eigenvalue with all other eigenvalues having negative real parts, hence the center manifold theory will be used to analyze the dynamics of the system near $\beta_1 = \beta^*$. Thus, E_0 is a non-hyperbolic equilibrium when $\beta_1 = \beta^*$.

Now, we calculate a right eigenvector $w = (w_1, w_2, w_3, w_4, w_5, w_6)^T$ of $J^*(E_0)$ associated with the zero eigenvalue.

$$\begin{bmatrix} -\mu & 0 & -\beta^* & 0 & -\beta_2 & 0 \\ 0 & -k_1 & \beta^* & 0 & \beta_2 & 0 \\ 0 & \gamma & -k_2 & 0 & \omega & 0 \\ 0 & 0 & \delta & -k_3 & 0 & 0 \\ 0 & 0 & 0 & \tau & -k_4 & 0 \\ 0 & 0 & 0 & \theta & \epsilon & -\mu \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We obtain the following system of equation

$$\begin{aligned} -\mu w_1 - \beta^* w_3 - \beta_2 w_5 &= 0 \\ -k_1 w_2 + \beta^* w_3 + \beta_2 w_5 &= 0 \\ \gamma w_2 - k_2 w_3 + \omega w_5 &= 0 \\ \delta w_3 - k_3 w_4 &= 0 \\ \tau w_4 - k_4 w_5 &= 0 \\ \theta w_4 + \varepsilon w_5 - \mu w_6 &= 0. \end{aligned} \tag{4.3.45}$$

Solving system of equation (4.3.45), we obtain

$$\begin{aligned} w_1 &= -\frac{(\beta^* k_3 k_4 + \beta_2 \tau \delta)}{\tau \delta \mu} w_5, \\ w_2 &= \frac{k_2 k_3 k_4 - \omega \tau \delta}{\gamma \tau \delta} w_5, \\ w_3 &= \frac{k_3 k_4}{\tau \delta} w_5, \\ w_4 &= \frac{k_4}{\tau} w_5, \\ w_5 &= w_5 > 0, \\ w_6 &= \frac{\theta k_4 + \epsilon \tau}{\tau \mu} w_5. \end{aligned}$$

The left eigenvector $v = (v_1, v_2, v_3, v_4, v_5, v_6)^T$ of $J^*(E_0)$ associated with the zero eigenvalue.

$$\begin{bmatrix} -\mu & 0 & 0 & 0 & 0 & 0 \\ 0 & -k_1 & \gamma & 0 & 0 & 0 \\ -\beta^* & \beta^* & -k_2 & \delta & 0 & 0 \\ 0 & 0 & 0 & -k_3 & \tau & \theta \\ -\beta_2 & \beta_2 & \omega & 0 & -k_4 & \epsilon \\ 0 & 0 & 0 & 0 & 0 & -\mu \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We obtain the following system of equation

$$\begin{aligned} -\mu v_1 &= 0 \\ -k_1 v_2 + \gamma v_3 &= 0 \\ -\beta^* v_1 + \beta^* v_2 - k_2 v_3 + \delta v_4 &= 0 \\ -k_3 v_4 + \tau v_5 + \theta v_6 &= 0 \\ -\beta_2 v_1 + \beta_2 v_2 + \omega v_3 - k_4 v_5 + \epsilon v_6 &= 0 \\ -\mu v_6 &= 0. \end{aligned} \tag{4.3.46}$$

Solving system of equation (4.3.46), we obtain

$$\begin{aligned} v_1 &= 0, \\ v_2 &= v_2 > 0, \\ v_3 &= \frac{k_1}{\gamma} v_2, \\ v_4 &= \frac{k_1 k_2 - \beta^* \gamma}{\delta \gamma} v_2, \\ v_5 &= \frac{k_3 (k_1 k_3 - \beta^* \gamma)}{\tau \delta \gamma} v_2, \\ v_6 &= 0. \end{aligned}$$

Since the first and sixth component of v are zero, we don't need the partial derivatives of f_1 and f_6. From the partial derivatives of f_2, f_3, f_4 and f_5 at the OG addiction free equilibrium point, the only non-zero second derivatives are in Table(4.3).

Table 4.3: Non-zero second derivatives

$\frac{\partial^2 f_2}{\partial x_3 \partial x_2} = \frac{\partial^2 f_2}{\partial x_2 \partial x_3} = \frac{\partial^2 f_2}{\partial x_4 \partial x_3} = \frac{\partial^2 f_2}{\partial x_3 \partial x_4} = \frac{\partial^2 f_2}{\partial x_5 \partial x_3} = \frac{\partial^2 f_2}{\partial x_3 \partial x_5} = \frac{\partial^2 f_2}{\partial x_6 \partial x_2} = \frac{\partial^2 f_2}{\partial x_2 \partial x_6} = \frac{-\beta^* \mu}{\Lambda}$
$\frac{\partial^2 f_2}{\partial^2 x_3} = \frac{-2\beta^* \mu}{\Lambda}, \frac{\partial^2 f_2}{\partial x_5 \partial x_2} = \frac{\partial^2 f_2}{\partial x_2 \partial x_5} = \frac{\partial^2 f_2}{\partial x_4 \partial x_5} = \frac{\partial^2 f_2}{\partial x_5 \partial x_4} = \frac{\partial^2 f_2}{\partial x_6 \partial x_5} = \frac{\partial^2 f_2}{\partial x_5 \partial x_6} = \frac{\partial^2 f_2}{\partial^2 x_5} = \frac{-\beta_2 \mu}{\Lambda}$
$\frac{\partial^2 f_2}{\partial x_3 \partial \beta_1} = 1$

The direction of the bifurcation at $R_0 = 1$ is determined by the signs of the bifurcation coefficients a and b. Hence,

$$
\begin{aligned}
a &= v_2 \sum_{i,j=1}^{6} w_i w_j \frac{\partial^2 f_2}{\partial x_i \partial x_j}(E_0) \\
a &= -v_2 \frac{\mu}{\Lambda}[2(\frac{k_2k_3k_4 - \omega\tau\delta}{\gamma\tau\delta}(\beta^*(\frac{k_3k_4}{\tau\delta} + \frac{\theta k_4 + \epsilon\tau}{\tau\mu}) + \beta_2) + 2\beta^* \frac{k_4k_3}{\tau\delta}(\frac{k_4}{\tau} + \frac{k_4k_3}{\tau\delta} + 1) \\
&\quad + \beta_2(2\frac{k_4}{\tau} + 2\frac{\theta k_4 + \epsilon\tau}{\tau\mu} + 1)]w_5^2 < 0.
\end{aligned}
$$

and

$$
\begin{aligned}
b &= v_2 \sum_{i=1}^{6} w_i \frac{\partial^2 f_2}{\partial x_i \partial \beta_1}(E_0) \\
b &= \frac{k_3k_4}{\tau\delta} w_5 v_2 > 0.
\end{aligned}
$$

Since $a < 0$ and $b > 0$ at $\beta_1 = \beta^*$. Therefore, based on the theorem (3.3.1), the system (4.2.1) exhibits forward bifurcation at $R_0 = 1$ and the unique endemic equilibrium is locally asymptotically stable for $R_0 > 1$. Which means the OG addiction-free equilibrium point and the endemic equilibrium point do not co-exist when $R_0 < 1$. From bifurcation analysis, we have already justified that the system (4.2.1) undergoes a forward bifurcation at $R_0 = 1$. Thus, R_0 plays an important role in the spread of OG addiction. If $R_0 < 1$, then it's easy to control the OG addiction. But when $R_0 > 1$, the community will experience endemic OG addiction spreading.

Using the numerical value model parameters(from Table (4.6)) the forward bifurcation diagram can be seen in Figure (4.3). Figure (4.3) show us when $R_0 < 1$, the system (4.3) has no endemic equilibrium point and the OG addiction-free equilibrium point is stable. When $R_0 > 1$, a stable endemic equilibrium appears and the OG addiction-free equilibrium becomes unstable, i.e., the exchange of stability of the equilibrium's (forward bifurcation) occurs at the bifurcation point $R_0^* = 1$.

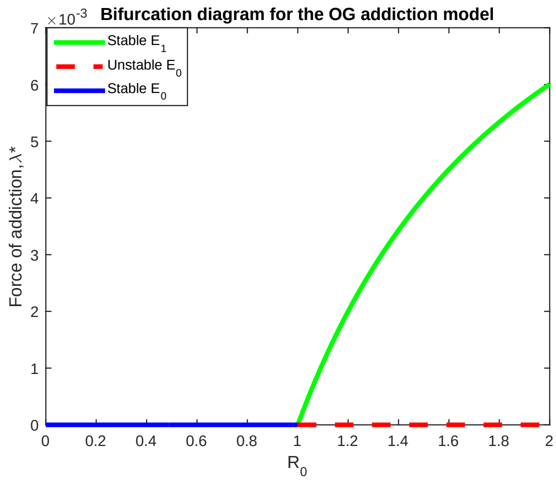


Figure 4.3: Forward bifurcation diagram for the OG addiction model (4.2.1).

4.4 Sensitivity analysis of the model

Sensitivity analysis for the basic reproduction number is being investigated to help determine the parameter value that contributes more to addiction transmission. In order to find out, we utilize the sensitivity index analysis by computing the relative sensitivity of R_0 based on Martcheva[36].

We compute the relative sensitivity of R_0 with respect to the model parameters $\beta_1, \beta_2, \gamma, \delta, \omega, \tau, \theta, \epsilon$ and μ (see definition(3.3.4)) using the expression $\Gamma_P^{R_0} = \frac{\partial R_0}{\partial p} \times \frac{p}{R_0}$, where p is the parameter of the model and R_0 is the quantity to be partiality differentiated.

a. The relative sensitivity of R_0 with respect to β_1 is given by

$$\begin{aligned}\Gamma_{\beta_1}^{R_0} &= \frac{\partial R_0}{\partial \beta_1} \times \frac{\beta_1}{R_0}\\ &= \frac{\beta_1(\tau+\theta\mu)(\omega+\epsilon+\mu)}{(\tau+\theta+\mu)(\omega+\epsilon+\mu)\beta_1+\delta\tau\beta_2} > 0.\end{aligned}$$

b. The relative sensitivity of R_0 with respect to β_2 is given by

$$\begin{aligned}\Gamma_{\beta_2}^{R_0} &= \frac{\partial R_0}{\partial \beta_2} \times \frac{\beta_2}{R_0}\\ &= \frac{\beta_2\delta\tau}{(\tau+\theta+\mu)(\omega+\epsilon+\mu)\beta_1+\delta\tau\beta_2} > 0.\end{aligned}$$

c. The relative sensitivity of R_0 with respect to ω is given by

$$\begin{aligned}\Gamma_{\omega}^{R_0} &= \frac{\partial R_0}{\partial \omega} \times \frac{\omega}{R_0}\\ &= \omega\frac{\tau\delta k_1(k_3\beta_1(\omega+\epsilon-(\tau+\theta))+\beta_2(\delta\tau-k_2k_3))}{(k_3k_4\beta_1+\delta\tau\beta_2)(k_1(k_2k_3k_4-\omega\delta\tau))} > 0.\end{aligned}$$

d. The relative sensitivity of R_0 with respect to γ is given by

$$\begin{aligned}\Gamma_{\gamma}^{R_0} &= \frac{\partial R_0}{\partial \gamma} \times \frac{\gamma}{R_0}\\ &= \frac{\mu}{\gamma+\mu} > 0.\end{aligned}$$

e. The relative sensitivity of R_0 with respect to τ is given by

$$\begin{aligned}\Gamma_{\tau}^{R_0} &= \frac{\partial R_0}{\partial \tau} \times \frac{\tau}{R_0}\\ &= \tau\frac{((k_4\beta_1+\delta\beta_2)(k_2k_3k_4-\tau\omega\delta)-(k_3k_4\beta_1+\tau\delta\beta_2)(k_2k_4-\omega\delta))}{(k_3k_4\beta_1+\delta\tau\beta_2)(k_1(k_2k_3k_4-\omega\delta\tau))}\\ &> 0.\end{aligned}$$

f. The relative sensitivity of R_0 with respect to θ is given by

$$\begin{aligned}\Gamma_{\theta}^{R_0} &= \frac{\partial R_0}{\partial \theta} \times \frac{\theta}{R_0} \\ &= -\theta \frac{\delta\tau(\delta+\mu)(\omega+\epsilon+\mu)}{(k_3k_4\beta_1+\delta\tau\beta_2)(k_1(k_2k_3k_4-\omega\delta\tau))} < 0.\end{aligned}$$

g. The relative sensitivity of R_0 with respect to δ is given by

$$\begin{aligned}\Gamma_{\delta}^{R_0} &= \frac{\partial R_0}{\partial \delta} \times \frac{\delta}{R_0} \\ &= -\delta \frac{[\beta_1(\tau+\theta+\mu)(\omega+\epsilon+\mu)-\tau(\omega+\beta_2\mu)]}{(k_3k_4\beta_1+\delta\tau\beta_2)(k_1(k_2k_3k_4-\omega\delta\tau))} \\ &\quad \times(\tau+\theta+\mu)(\omega+\epsilon+\mu) \\ &< 0.\end{aligned}$$

h. The relative sensitivity of R_0 with respect to ϵ is given by

$$\begin{aligned}\Gamma_{\epsilon}^{R_0} &= \frac{\partial R_0}{\partial \epsilon} \times \frac{\epsilon}{R_0} \\ &= -\epsilon \frac{\delta\tau(\gamma+\theta+\mu)(\omega\beta_1+\beta_2(\delta+\mu))}{\gamma(k_3k_4\beta_1+\delta\tau\beta_2)(k_1(k_2k_3k_4-\omega\delta\tau))} < 0.\end{aligned}$$

i. The relative sensitivity of R_0 with respect to μ is given by

$$\begin{aligned}\Gamma_{\mu}^{R_0} &= \frac{\partial R_0}{\partial \mu} \times \frac{\mu}{R_0} \\ &= -\mu \frac{(k_3k_4\beta_1+\delta\tau\beta_2)C-\beta_1(k_3+k_4)(k_1k_2(k_3k_4-\omega\delta\tau))}{(k_1(k_2k_3k_4-\omega\delta\tau))(k_3k_4\beta_1+\delta\tau\beta 2)} \\ &< 0,\end{aligned}$$

where

$$C = k_2(k_3k_4-\omega\delta\tau)+k_1(k_3k_4-\omega\delta\tau+k_2(k_3+k_4)).$$

Depending on the numerical values of the model parameters, let us verify the sensitivities of the parameters in the sensitivity indices table (4.4) and figure(4.10). Note that the sensitivity indices may depend on several parameters of the system but can also be constant, independent of any parameter.

Table 4.4: Numerical values of sensitivity indices of R_0

Parameters	Sensitivity indices	Values
β_1	+ve	0.0451
β_2	+ve	3.5348×10^{-5}
γ	+ve	0.1906
δ	-ve	-2.9622
τ	+ve	0.02302
ω	+ve	8.5008×10^{-4}
ε	-ve	-0.5478
θ	-ve	-0.0126
μ	-ve	-1.0030

4.5 Numerical simulations and discussion

In this section, we use numerical simulations to show the dynamical behavior of our model. The numerical simulations of the model of the system (4.2.1) carried out to illustrate graphically with the help of the ode45 Matlab tool. The sources of these parameters are mainly from literature and some are appropriate assumptions for simulation purpose. The aim of the simulation is to investigate the response of model parameters to OG addiction. Table (4.5) shows the initial condition values of state variables and Table (4.6) shows the set of parameter values that used in order to support the analytical results.

Table 4.5: The variable values of the modified model

Variables	$S(0)$	$E(0)$	$A(0)$	$T(0)$	$Q_i(0)$	$Q_p(0)$
Values	69000	34000	30000	6000	800	10200
Source	Assumed	Assumed	Assumed	Assumed	Assumed	Assumed

Table 4.6: The parameter values of the modified model

Parameters	Values	Source
Λ	1801.5	Assumed
β_1	0.553277	[33]
β_2	0.036	[63]
γ	0.051	[59]
δ	0.0028	Assumed
τ	0.002685	[33]
ω	0.04	[35]
ε	0.130325	[33]
θ	0.0612	[59]
μ	0.01201	[33]

We conducted a simulation study of the model system (4.2.1), and simulation graphs are given and interpreted as follows:

In Figure (4.4a) with R_0 =0.0121, we observe that for the basic reproduction number R_0 <1, all solution curves go to the OG addiction-free equilibrium point. As a result, the OG addiction goes extinct, and eradicated. In Figure (4.8), when R_0 <1 and taking different initial conditions for the state variables show, all trajectories of state variables go to their components of the OG addiction-free equilibrium point. These indicate that the OG addiction-free equilibrium point is globally asymptotically stable for the values of R_0 <1. This implies that whatever OG addiction persists in the population in the long run (after a certain period of time), the addiction will be eradicated from the population.

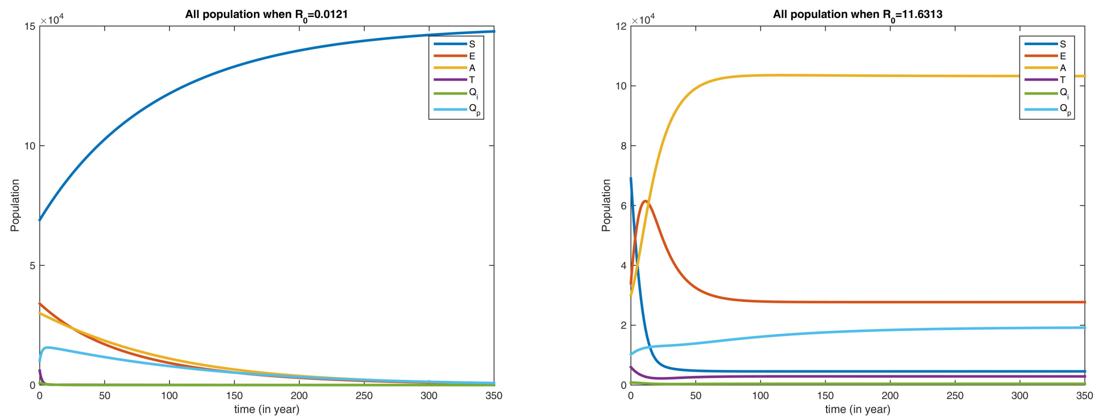
In Figure (4.4b) with R_0=11.6313, we observe that for the basic reproduction number $R_0 > 1$, all solution curves go away from the OG addiction-free equilibrium point. These indicate that the OG addiction-free equilibrium point is unstable for values of $R_0 > 1$, and the solutions will go to the endemic equilibrium point. Consequently, the OG addiction invades a population. In Figure (4.9) with $R_0 > 1$ and different initial conditions for the state variables, all trajectories of state variables go to their components of the endemic equilibrium point. These indicate that the endemic equilibrium point is globally asymptotically stable for the values of $R_0 > 1$.

Figure (4.10) and Table (4.4) shows that the sensitivity indexes of the basic reproduction number with regard to the model parameters. These parameters, which positive indices(β_1, β_2, γ, τ and ω) show that they have a great impact on the expansion of the gaming addiction in the community if their values increase(see figure (4.10a)-(4.10e)). The reason that the basic reproduction number increases is because their values increase, which means that the average number of secondary cases of OG addiction increases in the community. And also the parameters in which their sensitivity indices are negatives (δ, ϵ, θ and μ) have an influence on reducing at least the burden of the gaming addiction in the community; their values increase while the others are left constant. And also, as their values increase, the number of basic reproductions decreases(see figure (4.10g)-(4.10h)), which leads to minimizing the endemicity of the gaming addiction in the community. Therefore, with sensitivity analysis, one can gain insight into the appropriate intervention strategies to prevent and control the spread and behavior of the gaming addition described by model (4.2.1).

In figure (4.5), we observe that as the contact rate increases, the more individuals join the addicted compartment from the exposed class. The exposed population decrees significantly over time and addicted population increases. This is due to the fact the contact rate with addicted individuals increases, then the reproduction number increases, which means the addiction persists in the population. And it is realized that the population of addicted individuals will rise as the contact rate increases. Hence, to decrease the number of addicted individuals, one can decrease the contact rate.

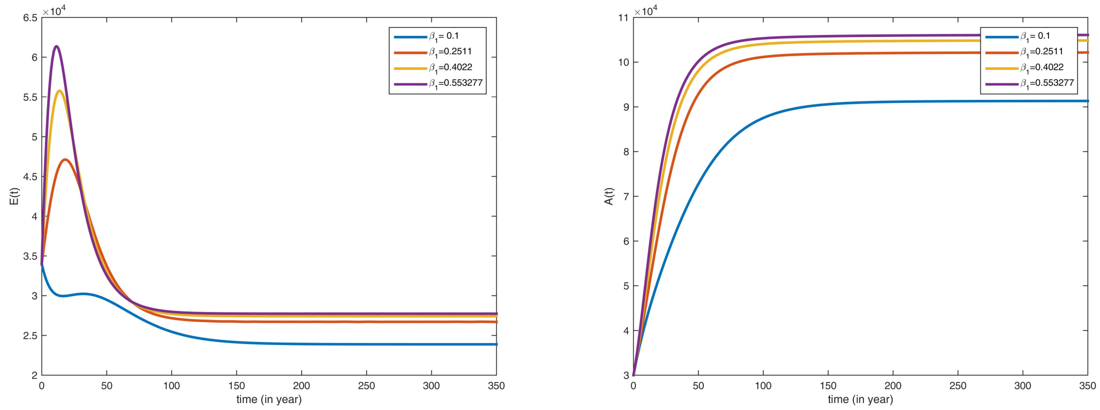
In figure (4.6), we observe that as the re-addiction rate increases, the addicted population increases while incompletely recovered population decrees. Because if the re-addiction rate of incompletely recovered individuals increases, then the reproduction number increases, which causes addiction to relapse and persist in the population. This shows that the high re-addiction rate of incompletely recovered individuals leads to an increase in the number of addicted individuals in the population. And it is realized that the population of addicted individuals will rise as the rate increases. This graph also demonstrates that the re-addiction rate of incompletely recovered individuals has a large impact on the relapse of the game addiction throughout the population. If the re-addiction rate is observed to be high, then the rate of addiction to the game will also be high, as would be expected logically. Hence, to decrease the number of secondary addictions, one can decrease the re-addiction rate of incompletely recovered individuals.

In figure (4.7), we observe that increasing the complete recovery rate of treated individuals minimizes the number of incomplete recovery of treated individuals while the number of completely recovered individuals increases. This is due to the fact that the population in the treatment compartment decreases as the complete recovery rate of treated individuals increases. This shows that the high complete recovery rate of treated individuals leads to an increase in the number of completely recovered individuals in the population. And it is realized that the population of completely recovered individuals will rise as the rate increases. Further, this graph demonstrates that the completely recovered rate of treated individuals has a large impact on the rate of incomplete recovery of treated individuals. Hence, to decrease the number of incomplete recoveries, one can increase the rate of complete recovery of treated individuals. To do that, addicted individuals should have commitment, and healthcare professionals should do their work properly as the work ethics require.



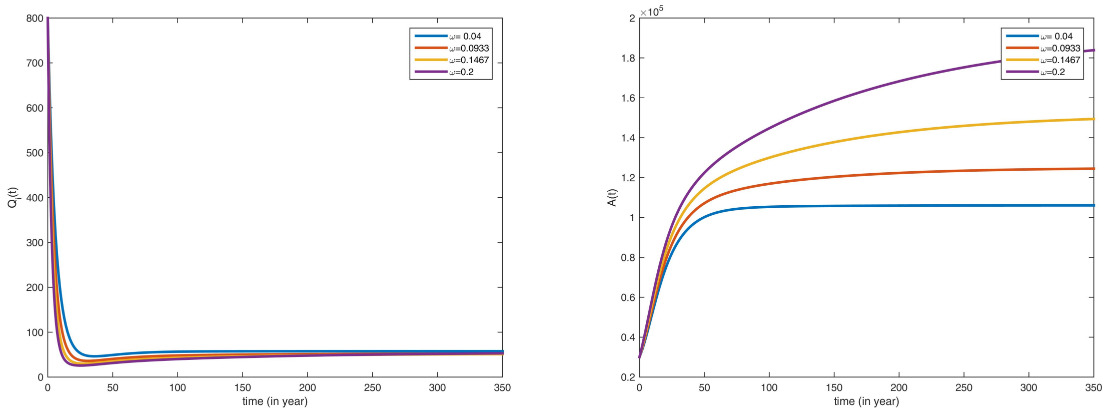
(a) Trajectories of state variables for R_0=0.0121 (b) Trajectories of state variables for R_0=11.6313

Figure 4.4: Trajectories of state for different R_0 value.



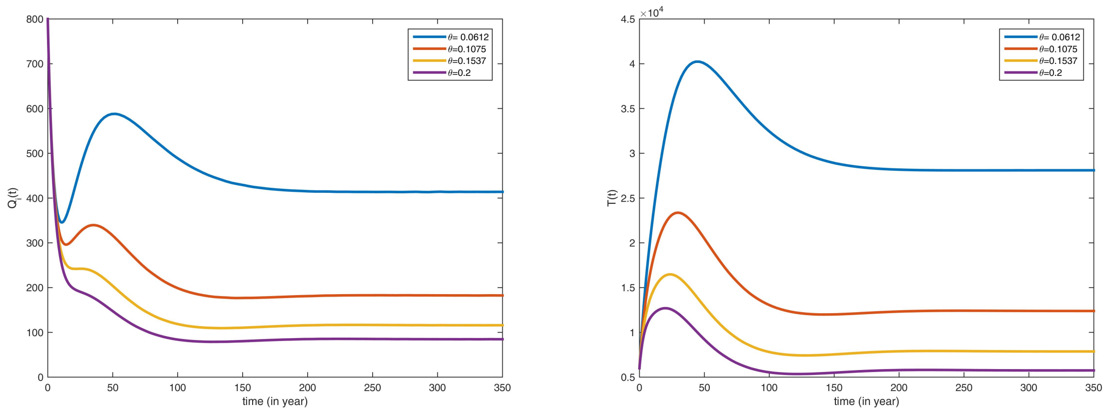
(a) (b)

Figure 4.5: Exposed and addicted population with varying contact rate (β_1).



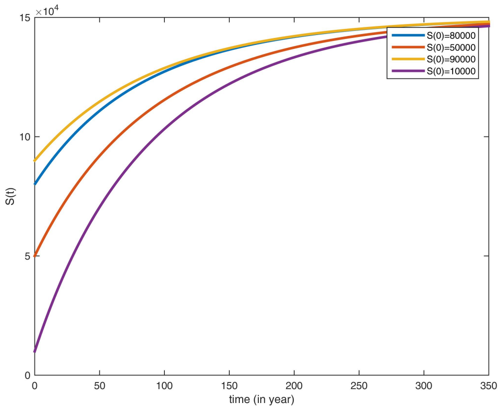
(a) (b)

Figure 4.6: Incompletely recovered and addicted population with varying re-addiction rate (ω).

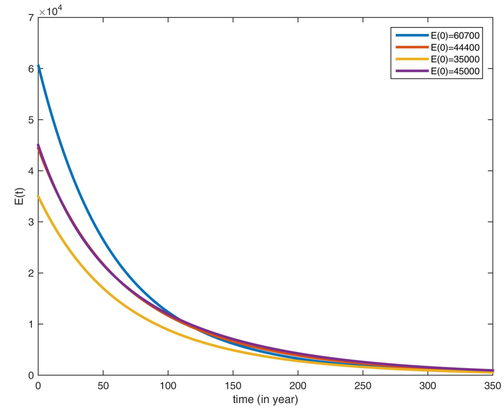


(a) (b)

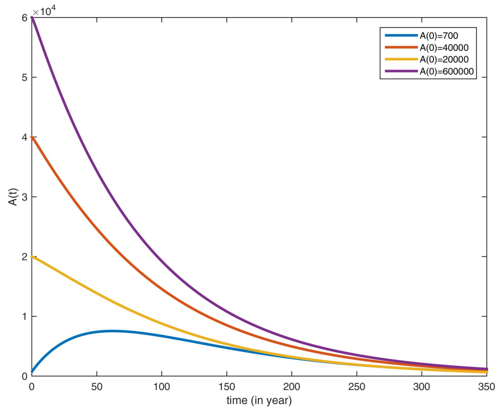
Figure 4.7: Incompletely recovered and treated population with varying complete recovery rate (θ).



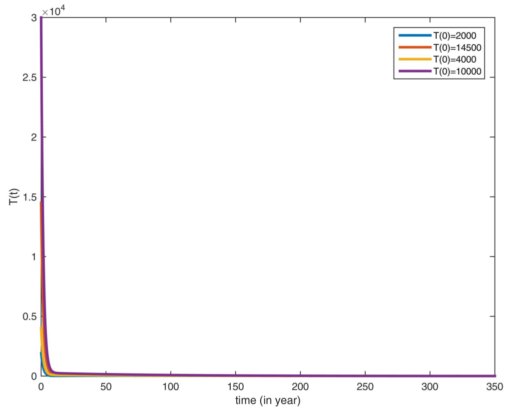
(a)



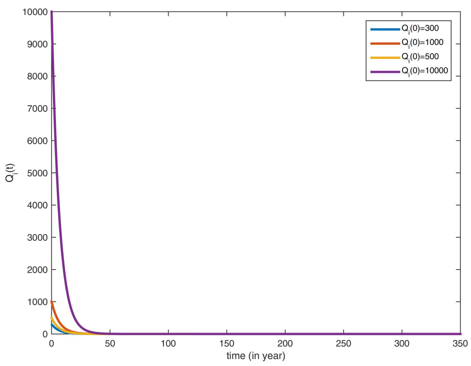
(b)



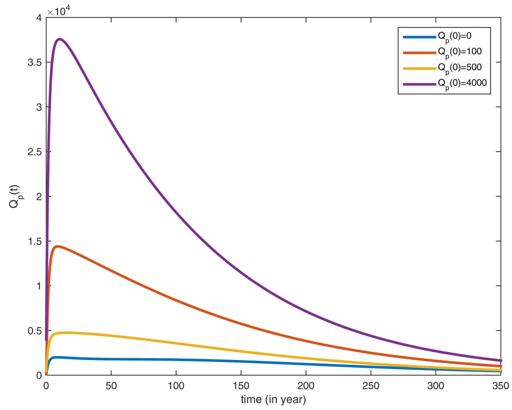
(c)



(d)



(e)



(f)

Figure 4.8: Trajectories of state variables for R_0=0.0121 with different initial conditions.

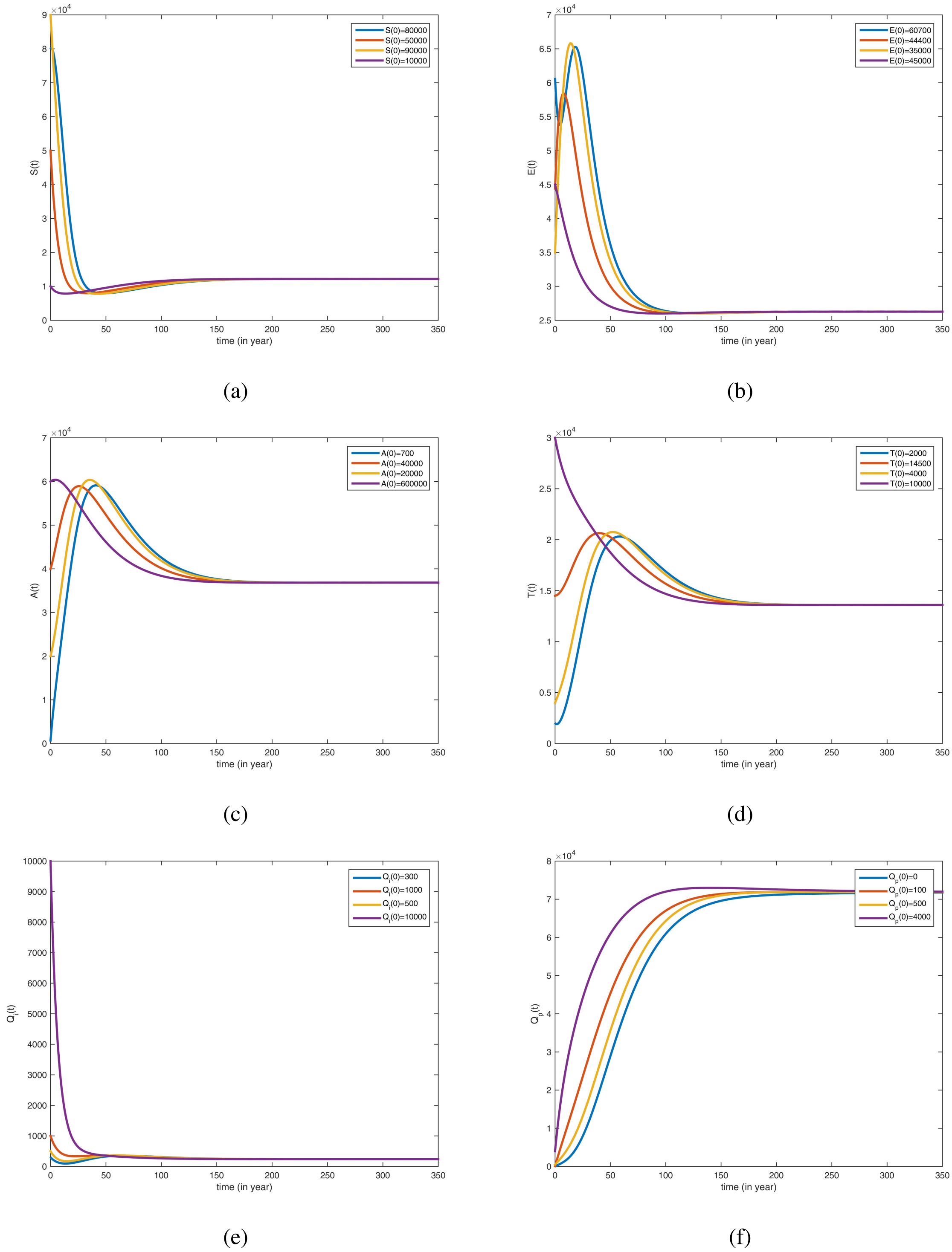
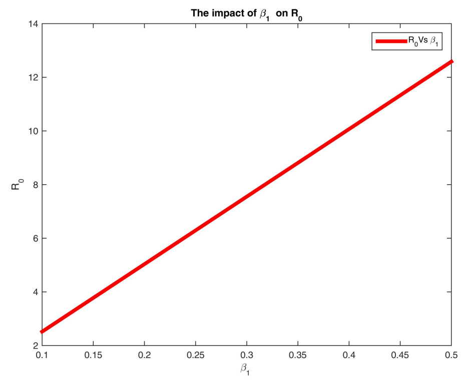
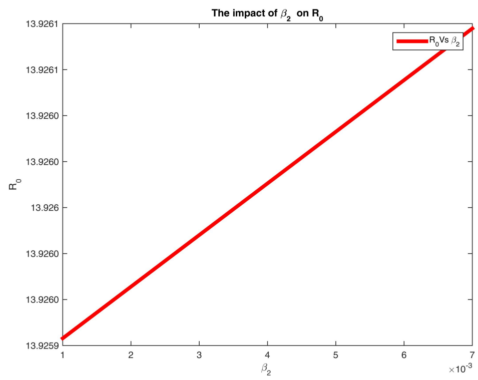


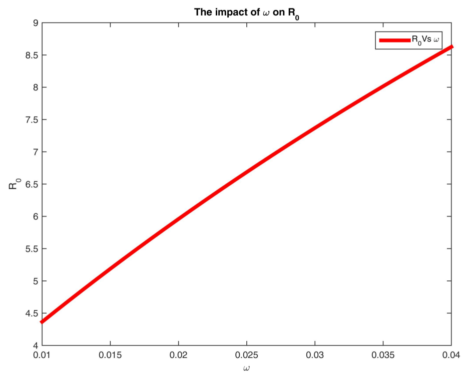
Figure 4.9: Trajectories of state variables for R_0=11.6313 with different initial conditions.



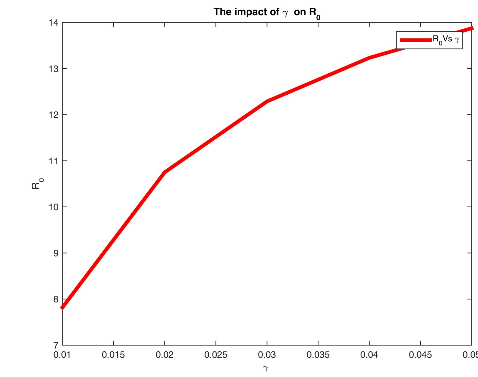
(a)



(b)



(c)



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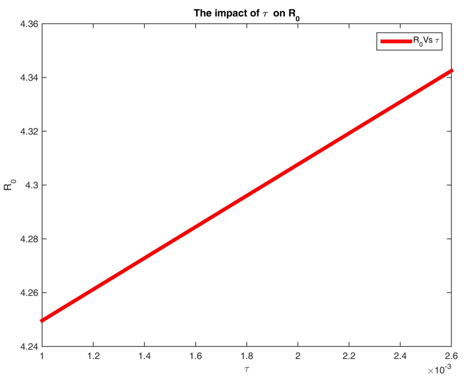
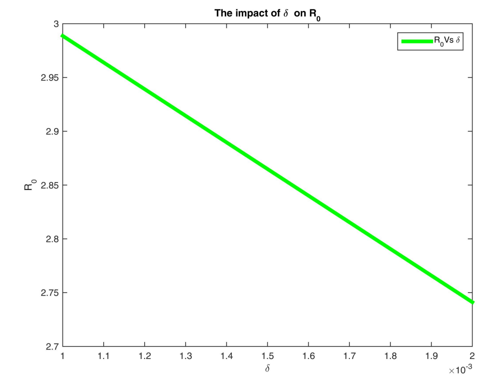
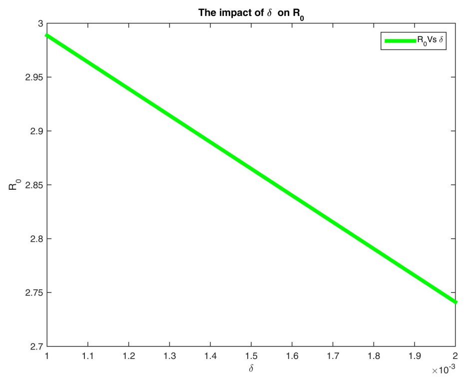


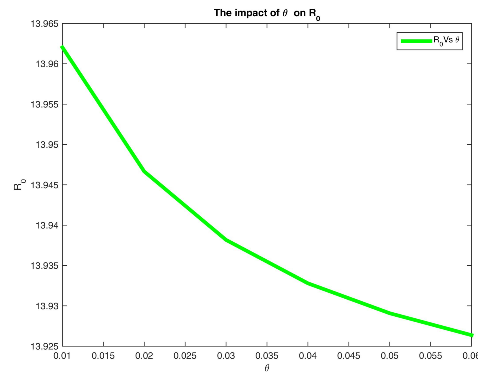
(e)



(f)



(g)



(h)

Figure 4.10: Impact of model parameters on R_0.

CHAPTER 5
OPTIMAL CONTROL ANALYSIS

In chapter (4), we presented a mathematical model for the transmission dynamics of OG addiction and studied well-posedness mathematically and epidemiologically without giving an intervention mechanism. In this chapter, the intervention mechanism is addressed using the theory of optimal control. Optimal control is a systematic mathematical framework in complex scenarios for decision-making. We utilize optimal control theory to find precautionary (for susceptible) and post-cautionary (for incompletely recovered) strategies that would minimize the total addicted population using optimum resources. The theory uses Pontryagin's Maximum Principle to minimize the cost by finding optimal strategies for controlling the system's parameters.

5.1 Formulation of optimal control problem

In this section, we extend model (4.2.1) to an optimal control model. In the model, we introduce two control functions, $u_1(t)$ and $u_2(t)$ in order to minimize the spread of OG addiction among the population and the associated costs. The variables are defined as the following: the fraction of precautionary activities (susceptible individuals to be informed about the impact and disadvantage of OG addiction through education camping) that helps to reduce the contact rate with addicted individuals represented by $u_1(t)$ and the fraction of post-cautionary activities (incompletely recovered individuals to be engaged in regular activities, giving special attention and restriction on game play) that helps to reduce the re-addiction rate represented by $u_2(t)$ with control set U:

$$U = \{(u_1(t), u_2(t)) : 0 \leq u_1(t), u_2(t) \leq \epsilon_i, 0 \leq t \leq t_f, 0 \leq \epsilon_i \leq 1, i = 1, 2\}, \qquad (5.1.1)$$

where $u_1(t)$ and $u_2(t)$ are Lebesgue measurable quantities bounded above by ϵ_i, which depends on the amount of resources available for the implementation of the control strategies.

The control variables u_i lie between 0 and 1, while u_i will depend on the amount of resources available to implement each of the control measures. If $u_1, u_2 = 0$, then no precaution and post-caution activity are done. This means the model (5.1.3) is uncontrolled, which is equivalent to the model system (4.2.1). If $u_1 = 1$ and $u_2 = 1$, it indicates precaution and post-caution activity are taken 100% (effectively and fruitfully) for susceptible populations and incompletely recovered individuals, respectively. In reality, this case is not possible. The control variables u_1 and u_2 are assumed to take values in $[0, 1)$ to eliminate the case. The primary objective of this section is to minimize the cost function

$$J(u_1(t), u_2(t)) = \int_0^{t_f} (K_1 E(t) + K_2 A(t) + \frac{1}{2}(C_1 u_1(t)^2 + C_2 u_2(t)^2))dt, \qquad (5.1.2)$$

where constants K_1, K_2, C_1 and C_2 are positive. The weight constants C_1 and C_2 are measures the cost or effort required for the implementation of each of the two control measures adopted while K_1 and K_2 are measures the relative importance of reducing the associated classes on the spread of the addiction. Finally, t_f denotes the terminal intervention time.

The goal is to find the optimal values of $u_1(t)$ and $u_2(t)$ that minimize the objective function(5.1.2) while taking into consideration the state system(5.1.3).

$$\begin{aligned}
\frac{dS}{dt} &= \Lambda - \frac{((1-u_1)\beta_1 A + \beta_2 Q_i)S}{N} - \mu S,\\
\frac{dE}{dt} &= \frac{((1-u_1)\beta_1 A + \beta_2 Q_i)S}{N} - (\gamma+\mu)E,\\
\frac{dA}{dt} &= \gamma E + (1-u_2)\omega Q_i - (\delta+\mu)A,\\
\frac{dT}{dt} &= \delta A - (\tau+\theta+\mu)T,\\
\frac{dQ_i}{dt} &= \tau T - ((1-u_2)\omega + \varepsilon + \mu)Q_i,\\
\frac{dQ_p}{dt} &= \varepsilon Q_i + \theta T - \mu Q_p.
\end{aligned} \tag{5.1.3}$$

with initial conditions

$$S(0) > 0, E(0) \geq 0, A(0) \geq 0, Q_i(0) \geq 0, Q_p(0) \geq 0.$$

5.2 Existence and characterization of the optimal control

In this section, we investigate the existence of an optimal control for our model (5.1.3). The existence of the optimal control can be shown by using the approach of Fleming and Rishel [14]. The boundedness of the solution of the system (5.1.3) for a finite time interval is needed to establish the existence of an optimal control and the uniqueness of the optimality system. For the model system (5.1.3) to be epidemiologically meaningful, it is important to prove that all its state variables are positive for all time. In order to establish the upper bounds for the solutions, we consider an equation for the total population size, N. The rate change of the total population, obtained by adding all the equations in the model (5.1.3), is given by

$$\frac{dN(t)}{dt} = \frac{dS(t)}{dt} + \frac{dE(t)}{dt} + \frac{dA(t)}{dt} + \frac{dT(t)}{dt} + \frac{dQ_i(t)}{dt} + \frac{dQ_p(t)}{dt},$$

$$\frac{dN(t)}{dt} = \Lambda - \mu N(t). \tag{5.2.1}$$

Re-arranging (5.2.1) yields

$$\frac{dN(t)}{dt} + \mu N(t) = \Lambda. \tag{5.2.2}$$

Since (5.2.2) is a linear first-order differential equation, finding the integrating factor, yields $e^{\mu t}$. Multiplying both sides of the equation (5.2.2) by $e^{\mu t}$ and integrating both sides, we obtain

$$N(t) \leq \frac{\Lambda}{\mu} + N(0)e^{-\mu t}. \quad (5.2.3)$$

As $t \to \infty$, then $N(t) \to \frac{\Lambda}{\mu}$. Thus, $N(t)$ is bounded above by $\frac{\Lambda}{\mu}$. The upper bound for $N(t)$ is also the upper bound for $S(t), E(t), A(t), T(t), Q_i(t), Q_p(t)$ since they are positive from our assumption, as we proved in the theorem (4.3.2).

We have already justified the boundedness of the solution of the basic OG addiction model (4.2.1), the optimal control system (5.1.3), and all the state variables involved in the model are continuously differentiable.

Theorem 5.2.1. (**Existence of optimal control solution**) Given the objective function $J(u_1(t), u_2(t))$ (5.1.2) with admissible control set U subject to the system (5.1.3), then there exist an optimal control $u^* = (u_1^*, u_2^*) \in U$ such that

$$J(u_1^*, u_2^*) = \min_{(u_1^*, u_2^*)} J(u_1(t), u_2(t)). \quad (5.2.4)$$

Proof. To prove the existence of optimal control, we need to verify the following conditions.

(i) The control set U is non empty set.
We use proof by contradiction. It is clear that giving control can realize the objective function. Suppose we consider the following objective function:

$$maxJ(\vec{U}) = \int_0^{t_f} (K_1E(t) + K_2A(t) + \frac{1}{2}(C_1u_1(t)^2 + C_2u_2(t)^2))dt.$$

This implies that the aim of the objective function is to maximize the exposed and addicted population. On the other hand, we have that the range $t = [0, t_f]$ is bounded, i.e., there is a process to limit the addiction. Then, the control variable must be a minimum of

$$minJ(\vec{U}) = \int_0^{t_f} (K_1E(t) + K_2A(t) + \frac{1}{2}(C_1u_1(t)^2 + C_2u_2(t)^2))dt$$

and proved that control is not an empty set.

(ii) The set U is convex and closed.

a) Assume that $k \in U$ and $k' \in U$. For any $\eta \in [0, 1]$, it will be shown that $n = \eta k + (1 - \eta)k' \in U$. It is obvious that we obtain $\eta k + (1 - \eta)k' \leq 1$ if $\eta k \leq \eta$ and $(1 - \eta)k' \leq (1 - \eta)$. Finally, we obtained $0 \leq \eta k + (1 - \eta)k' \leq 1$, $\forall k, k' \in U$, and $\forall \eta \in [0, 1]$. Then the control U is a convex set.

b) Assuming that the control variable $k \notin [x, y]$, it can indicate that $k < x$ or $k > y$. Now if $k < x$, then $\epsilon_k = |k - x| > 0$ exists, meaning that an $\emptyset$, $[x, y] \cap V_\epsilon(k) = \emptyset$, is the intersection of the set and neighborhood of control. If $k > y$, then $\epsilon_k = |k - y| > 0$ exist, so that $[x, y] \cap V_\epsilon(k) = \emptyset$ is intersection of the set the neighborhood of control. Consequently, the control k is a closed set, where $k \in U$.

(iii) The right-hand equation of the system (5.1.3) is bounded by some control design and linear function.

We re-write the system (5.1.3) in matrix form as follow

$$\begin{bmatrix} \frac{dS}{dt} \\ \frac{dE}{dt} \\ \frac{dA}{dt} \\ \frac{dT}{dt} \\ \frac{dQ_i}{dt} \\ \frac{dQ_p}{dt} \end{bmatrix} = \begin{bmatrix} -\beta_2\frac{Q_iS}{N} - \mu S \\ \beta_2\frac{Q_iS}{N} - (\gamma+\mu)E \\ \gamma E - (\delta+\mu)A \\ \delta A - (\tau+\theta+\mu)T \\ \tau T - (\varepsilon+\mu)Q_i \\ \varepsilon Q_i + \theta T - \mu Q_p \end{bmatrix} + \begin{bmatrix} -(1-u_1)\beta_1\frac{AS}{N} \\ (1-u_1)\beta_1\frac{AS}{N} \\ (1-u_2)\omega Q_i \\ 0 \\ -(1-u_2)\omega Q_i \\ 0 \end{bmatrix} + \begin{bmatrix} \Lambda \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ \beta_2\frac{Q_iS}{N} \\ \gamma E \\ \delta A \\ \tau T \\ \varepsilon Q_i + \theta T \end{bmatrix} - \begin{bmatrix} \beta_2\frac{Q_iS}{N} + \mu S \\ (\gamma+\mu)E \\ (\delta+\mu)A \\ (\tau+\theta+\mu)T \\ (\varepsilon+\mu)Q_i \\ \mu Q_p \end{bmatrix} + \begin{bmatrix} 0 \\ (1-u_1)\beta_1\frac{AS}{N} \\ (1-u_2)\omega Q_i \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$- \begin{bmatrix} (1-u_1)\beta_1\frac{AS}{N} \\ 0 \\ 0 \\ 0 \\ (1-u_2)\omega Q_i \\ 0 \end{bmatrix} + \begin{bmatrix} \Lambda \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$< \begin{bmatrix} 0 \\ \beta_2\frac{Q_iS}{N} \\ \gamma E \\ \delta A \\ \tau T \\ \varepsilon Q_i + \theta T \end{bmatrix} + \begin{bmatrix} 0 \\ (1-u_1)\beta_1\frac{AS}{N} \\ (1-u_2)\omega Q_i \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \Lambda \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\leq \left\| \begin{bmatrix} 0 \\ \beta_2\frac{Q_iS}{N} \\ \gamma E \\ \delta A \\ \tau T \\ \varepsilon Q_i + \theta T \end{bmatrix} \right\| + \left\| \begin{bmatrix} 0 \\ (1-u_1)\beta_1\frac{AS}{N} \\ (1-u_2)\omega Q_i \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\| + \left\| \begin{bmatrix} \Lambda \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\|$$

$$\leq \vec{K} = \begin{bmatrix} K_1 \\ K_2 \\ K_3 \\ K_4 \\ K_5 \\ K_6 \end{bmatrix}$$

Since the right-hand equation of the system (5.1.3) is bounded by some control design and linear function, condition (iii) is proved.

(iv) The integral

$$L(E, A, u_1, u_2, t) = K_1E(t) + K_2A(t) + \frac{1}{2}(C_1u_1(t)^2 + C_2u_2(t)^2),$$

is convex on with respect to the parameters u_1 and u_2 and $L(E, A, u_1, u_2, t) \geq q(u)$ where $q(u)$ is continuous and $||u||^{-1}q(u) \to \infty$ when $||u|| \to \infty$. Here $u = (u_1, u_2)$.

To show that the integral of the cost function($L(E, A, u_1, u_2, t)$) is convex, is sufficient to show that $w(t, u)$ is convex with respect to control parameters u_1, u_2. Hence, $w(t, u) = \frac{1}{2}(C_1u_1(t)^2 + C_2u_2(t)^2)$ is a finite linear combination with positive coefficients of the functions $w_1(t, u) = \frac{1}{2}C_1u_1(t)^2$ and $w_2(t, u) = \frac{1}{2}C_2u_2(t)^2$.
We show that $w : U \to \mathbb{R}^+$ defined by $w(u) = \frac{1}{2}||u||^2$ is convex. To show this, let $p_1, p_2 \in U$. Then for any $\pi \in [0, 1]$

$$\begin{aligned}
&w(\pi p_1 + (1-\pi)p_2) - (\pi w(p_1) + (1-\pi)w(p_2))\\
=&\frac{1}{2}[||\pi p_1 + (1-\pi)p_2||^2 - \pi||p_1||^2 - (1-\pi)||p_2||^2]\\
\leq&\frac{1}{2}[\pi^2||p_1||^2 + 2\pi(1-\pi)||p_1||||p_2|| + (1-\pi)^2||p_2||^2 - \pi||p_1||^2\\
&-(1-\pi)||p_2||^2]\\
=&\frac{1}{2}[\pi(\pi-1)||p_1||^2 + 2\pi(\pi-1)||p_1||||p_2|| + \pi(\pi-1)||p_2||^2]\\
=&\frac{1}{2}\pi(\pi-1)[||p_1||^2 - 2||p_1||||p_2|| + ||p_2||^2]\\
=&\frac{1}{2}\pi(\pi-1)(||p_1|| - ||p_2||)^2 \leq 0,
\end{aligned}$$

since $\pi \in [0, 1]$.
$\implies w(\pi p_1 + (1-\pi)p_2) \leq \pi w(p_1) + (1-\pi)w(p_2)$. Therefor, the integrand objective function $L(E, A, u_1, u_2, t)$ is convex with respect to $u_1(t), u_2(t)$.

Additionally

$$\begin{aligned}
L(E, A, u_1, u_2, t) &= K_1E(t) + K_2A(t) + \frac{1}{2}(C_1u_1(t)^2 + C_2u_2(t)^2)\\
&\geq \frac{1}{2}(C_1u_1(t)^2 + C_2u_2(t)^2).
\end{aligned}$$

We define a continuous function $q(u) = \pi||u||^2$, where $\pi = min(\frac{K_1}{2}, \frac{K_2}{2}) > 0$ and $u = (u_1(t), u_2(t))$.

Then we have

$$L(E, A, u_1, u_2, t) \geq \frac{1}{2}(C_1u_1(t)^2 + C_2u_2(t)^2) \geq \pi||u||^2, \tag{5.2.5}$$

since $\pi = min(\frac{K_1}{2}, \frac{K_2}{2}) > 0$. This implies that $L(E, A, u_1, u_2, t) \geq q(u)$.
Consider, $||u||^{-1}q(u) = ||u||^{-1}\pi||u||^2 = \pi||u||$. This gives that $||u||^{-1}q(u) = \pi||u|| \to \infty$ when $||u|| \to \infty$. Thus, conduction (iv) is proved.

Hence, all conduction's (i)-(iv) shows that there exist an optimal control $u^* = (u_1, u_2)$ that minimizes the cost function $J(u_1(t), u_2(t))$ over U. Therefore, the existence of optimal control is established.

□

5.3 The Hamiltonian and Optimality system

We used Pontryangin's Maximum Principle (3.4.1) based on the approach in ([1],[25],[32],-[51]) to drive the necessary conditions that an optimal control must satisfy. The optimal control and the state are found by solving the following optimality system

(a) The state system (5.1.3)

(b) The adjoint system

(c) Tranversality conditions and

(d) The characterization of the optimal control u_i^*

Pontryangin's maximum principle converts the objective functional (5.1.2) subject to the state system (5.1.3) into a problem of minimizing point-wise a Hamiltonian ($\mathbb{H}$), with respect to u_1 and u_2 as:

$$\begin{aligned}
\mathbb{H} =& K_1E + K_2A + \frac{1}{2}(C_1u_1^2 + C_2u_2^2)\\
&+\lambda_1(\Lambda - \frac{((1-u_1)\beta_1A + \beta_2Q_i)S}{N} - \mu S)\\
&+\lambda_2(\frac{((1-u_1)\beta_1A + \beta_2Q_i)S}{N} - (\gamma+\mu)E)\\
&+\lambda_3(\gamma E + (1-u_2)\omega Q_i - (\delta+\mu)A)\\
&+\lambda_4(\delta A - (\tau+\theta+\mu)T)\\
&+\lambda_5(\tau T - ((1-u_2)\omega + \varepsilon + \mu)Q_i)\\
&+\lambda_6(\varepsilon Q_i + \theta T - \mu Q_p).
\end{aligned} \tag{5.3.1}$$

where $\lambda_i, i = 1,2,3,4,5,6$ represent the adjoint variables associated with the state variables S, E, A, T, Q_i and Q_p to be determined suitably by applying Pontryagin's Maximal Principle (3.4.1).

Theorem 5.3.1. (**Necessary optimality conditions**) For an optimal control set u_1, u_2 that minimizes J over U, there are adjoint variables $\lambda_1, \lambda_2, ..., \lambda_6$ such that;

$$
\begin{aligned}
\frac{d\lambda_1}{dt} &= \lambda_1((1-u_1)\beta_1\frac{A}{N} + \beta_2\frac{Q_i}{N} + \mu) - \lambda_2((1-u_1)\beta_1\frac{A}{N} + \beta_2\frac{Q_i}{N}),\\
\frac{d\lambda_2}{dt} &= -K_1 + \lambda_2(\gamma+\mu) - \lambda_3\gamma,\\
\frac{d\lambda_3}{dt} &= -K_2 + (\lambda_1-\lambda_2)(1-u_1)\beta_1\frac{S}{N} + \lambda_3(\delta+\mu) - \lambda_4\delta,\\
\frac{d\lambda_4}{dt} &= \lambda_4(\tau+\mu+\theta) - \lambda_5\tau - \lambda_6\theta,\\
\frac{d\lambda_5}{dt} &= (\lambda_1-\lambda_2)\beta_2\frac{S}{N} - \lambda_3(1-u_2)\omega + \lambda_5((1-u_2)\omega+\varepsilon+\mu) - \lambda_6\varepsilon,\\
\frac{d\lambda_6}{dt} &= \mu\lambda_6.
\end{aligned}
\tag{5.3.2}
$$

with transversal conditions

$$\lambda_i(t_f) = 0, i = 1,2,...,6. \tag{5.3.3}$$

Moreover, we obtain the control set (u_1, u_2) characterized by

$$
\begin{aligned}
u_1^* &= max\{0, min\{\epsilon_1, \frac{\beta_1(\lambda_2-\lambda_1)AS}{NC_1}\}\},\\
u_2^* &= max\{0, min\{\epsilon_2, \frac{\omega Q_i(\lambda_3-\lambda_4)}{C_2}\}\}.
\end{aligned}
\tag{5.3.4}
$$

Proof. The form of the adjoint equation and transversality conditions are standard results from Pontryagin's Maximum Principle (3.4.1) following the approach in ([1],[25],[32]). We differentiate Hamiltonian (5.3.1) with respect to the state variables S, E, A, T, Q_i and Q_p respectively, and then the adjoint system can be written as

$$
\begin{aligned}
\frac{d\lambda_1}{dt} &= \lambda_1((1-u_1)\beta_1\frac{A}{N} + \beta_2\frac{Q_i}{N} + \mu) - \lambda_2((1-u_1)\beta_1\frac{A}{N} + \beta_2\frac{Q_i}{N}),\\
\frac{d\lambda_2}{dt} &= -K_1 + \lambda_2(\gamma+\mu) - \lambda_3\gamma,\\
\frac{d\lambda_3}{dt} &= -K_2 + (\lambda_1-\lambda_2)(1-u_1)\beta_1\frac{S}{N} + \lambda_3(\delta+\mu) - \lambda_4\delta,\\
\frac{d\lambda_4}{dt} &= \lambda_4(\tau+\mu+\theta) - \lambda_5\tau - \lambda_6\theta,\\
\frac{d\lambda_5}{dt} &= (\lambda_1-\lambda_2)\beta_2\frac{S}{N} - \lambda_3(1-u_2)\omega + \lambda_5((1-u_2)\omega+\varepsilon+\mu) - \lambda_6\varepsilon,\\
\frac{d\lambda_6}{dt} &= \mu\lambda_6.
\end{aligned}
$$

with transversal conditions

$$\lambda_i(t_f) = 0, i = 1,2,...,6.$$

Similarly by following the approach of Pontryagin et al.[51], the characterization of optimal controls u_1^*, u_2^*, i.e, the optimality equations are obtained based on the conditions: $\frac{\partial H}{\partial u_1}$ and $\frac{\partial H}{\partial u_2}$, which gives,

$$\frac{\partial H}{\partial u_1} = C_1 u_1 + \lambda_1 \frac{\beta_1 AS}{N} - \lambda_2 \frac{\beta_2 AS}{N}, \tag{5.3.5}$$

$$\frac{\partial H}{\partial u_2} = C_2 u_2 - \lambda_3 \omega Q_i + \lambda_5 \omega Q_i. \tag{5.3.6}$$

Using the optimality condition(i.e. setting equitation (5.3.5) and (5.3.6) equals to zero) we obtain,

$$C_1 u_1 + \lambda_1 \frac{\beta_1 AS}{N} - \lambda_2 \beta_2 \frac{AS}{N} = 0,$$

$$\implies u_1 = \frac{\beta_1(\lambda_2 - \lambda_1)AS}{NC_1}. \tag{5.3.7}$$

$$C_2 u_2 - \lambda_3 \omega Q_i + \lambda_5 \omega Q_i = 0,$$

$$\implies u_2 = \frac{\omega Q_i(\lambda_3 - \lambda_5)}{C_2}. \tag{5.3.8}$$

Since u_1 and u_2 are bounded in U by ϵ_1 and ϵ_2, respectively, optimal control becomes;

$$\begin{aligned} u_1^* &= max\{0, min\{\epsilon_1, \frac{\beta_1(\lambda_2 - \lambda_1)AS}{NC_1}\}\}, \\ u_2^* &= max\{0, min\{\epsilon_2, \frac{\omega Q_i(\lambda_3 - \lambda_5)}{C_2}\}\}. \end{aligned} \tag{5.3.9}$$

□

The optimality system is formed from the state system (5.1.3) and the adjoint variable system(5.3) by incorporating the characterized control set and initial and transversal condition. Then, we have the following optimality system

$$
\begin{aligned}
\frac{dS}{dt} &= \Lambda - \frac{((1-u_1)\beta_1 A + \beta_2 Q_i)S}{N} - \mu S,\\
\frac{dE}{dt} &= \frac{((1-u_1)\beta_1 A + \beta_2 Q_i)S}{N} - (\gamma+\mu)E,\\
\frac{dA}{dt} &= \gamma E + (1-u_2)\omega Q_i - (\delta+\mu)A,\\
\frac{dT}{dt} &= \delta A - (\tau+\mu)T - \theta T,\\
\frac{dQ_i}{dt} &= \tau T - ((1-u_2)\omega + \varepsilon + \mu)Q_i,\\
\frac{dQ_p}{dt} &= \varepsilon Q_i + \theta T - \mu Q_p,\\
\frac{d\lambda_1}{dt} &= \lambda_1((1-u_1)\beta_1\frac{A}{N} + \beta_2\frac{Q_i}{N} + \mu) - \lambda_2((1-u_1)\beta_1\frac{A}{N} + \beta_2\frac{Q_i}{N}),\\
\frac{d\lambda_2}{dt} &= -K_1 + \lambda_2(\gamma+\mu) - \lambda_3\gamma,\\
\frac{d\lambda_3}{dt} &= -K_2 + (\lambda_1-\lambda_2)(1-u_1)\beta_1\frac{S}{N} + \lambda_3(\delta+\mu) - \lambda_4\delta,\\
\frac{d\lambda_4}{dt} &= \lambda_4(\tau+\mu+\theta) - \lambda_5\tau - \lambda_6\theta,\\
\frac{d\lambda_5}{dt} &= (\lambda_1-\lambda_2)\beta_2\frac{S}{N} - \lambda_3(1-u_2)\omega + \lambda_5((1-u_2)\omega+\varepsilon+\mu) - \lambda_6\varepsilon,\\
\frac{d\lambda_6}{dt} &= \mu\lambda_6.
\end{aligned}
$$

$$
\lambda_i, i = 1, 2, .., 6,
$$
$$
(S(0), E(0), A(0), T(0), Q_i(0), Q_p(0)) = (S_0, E_0, A_0, T_0, Q_{i0}, Q_{p0}).
$$

5.4 Numerical simulations and discussion

In this section, we compare system (4.2.1) with our model with optimum control system (5.1.3) using numerical solutions to illustrate its dynamic behavior. We compare the exposed and addicted populations with each control strategy and without control, and then we assess the OG cost for each strategy. The variable and parameter values from Tables (4.5) and (4.6) are used in the simulation, in addition to the extra coefficient values shown in Table (5.1). Additionally, we examine various strategies meant to eradicate and control OG addiction within the population:

(A) **Strategy A:** Precautionary activity($u_1 \neq 0$ and $u_2 = 0$)

(B) **Strategy B:** Post-cautionary activity($u_1 = 0$ and $u_2 \neq 0$)

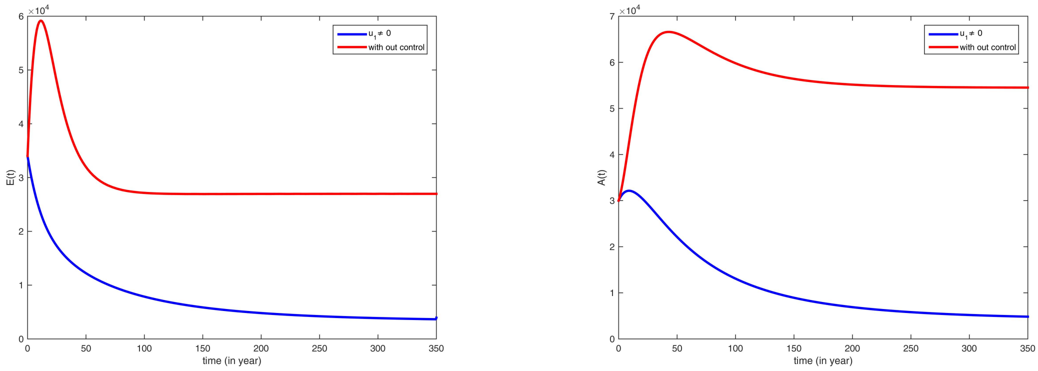
(C) **Strategy C:** Combination of strategies (A) and (B) ($u_1 \neq 0$ and $u_2 \neq 0$)

Table 5.1: The value of the coefficients under the objective function

Parameters	C_1	C_2	K_1	K_2
Values	9	9	2	2
Source	Assumed	Assumed	Assumed	Assumed

A. Control with precautionary activity only

Under this strategy, we control the contact rate with precautionary activity only for the exposed and addicted population and compere applying strategy (A) and without optimal control. In figure (5.1), minimizing the contact rate with addicted individuals (β_1) yields exposed (figure 5.1a) and addicted (figure (5.1b)) populations showing a significant drop when there is control compared to situations with no control. Figure (5.1) shows that applying strategies like education campaigns to OG addiction can reduce the probability of individuals becoming more exposed and addicted.



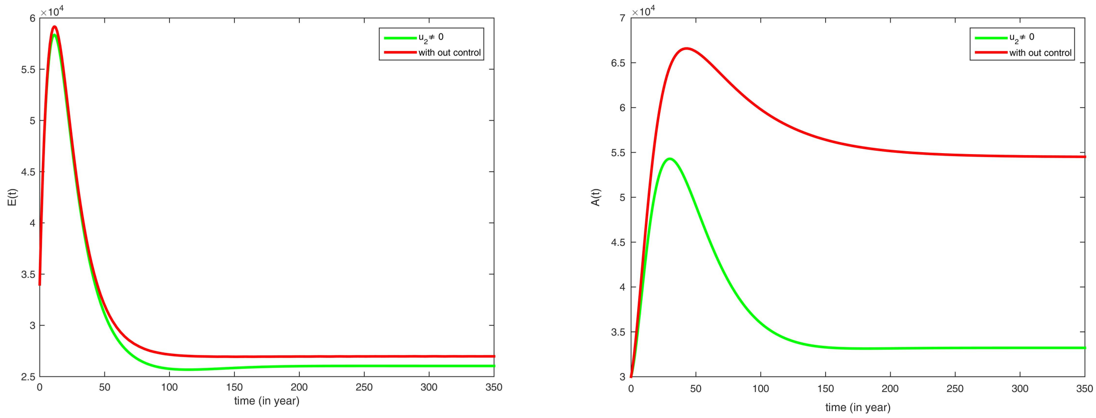
(a)

(b)

Figure 5.1: The effect of applying strategy (A).

B. Control with post-cautionary activity only

Under this strategy, we compere exposed and addicted population applying strategy (B) and without optimal control. In figure (5.2), minimizing the re-addiction rate of incompletely recovered individuals(ω) yields an exposed population (figure (5.2a)) and an addicted population (figure (5.2b)), which shows a drop using the strategy as compared with no control. Here, figure (5.2) shows us that applying strategies like restriction on game play and engaging incompletely recovered individuals in regular activities can reduce the probability of individuals becoming exposed and addicted. But figure (5.2a) shows us the number of exposed individuals drope is not significant when we compare with no control strategy applied. It requires a pre-work or additional/combined strategy to drop significantly.



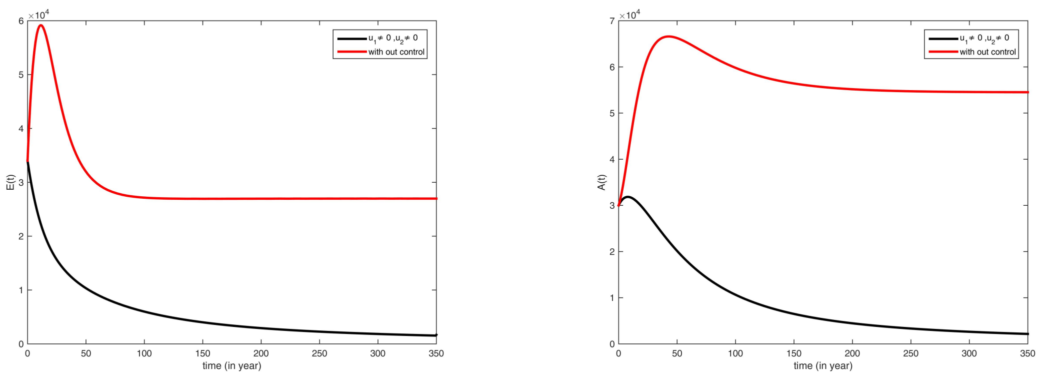
(a) (b)

Figure 5.2: The effect of applying strategy (B).

C. Control with precautionary and post-cautionary activity

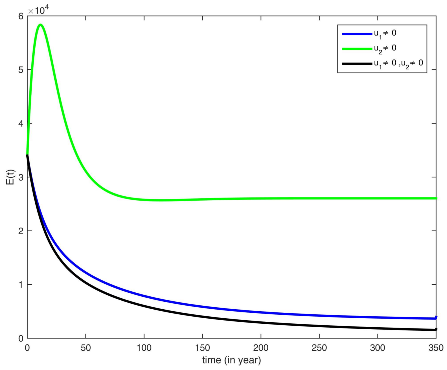
Lastly, we present the effects of implementing the combined strategies (strategy (C)). The corresponding simulation results are illustrated in figure (5.3). Exposed population(figure (5.3a)) and addicted population(figure (5.3b)) shows a drop. From figure (5.4), one can easily see that the number of exposed individuals (figure (5.4a)) and addicted individuals (figure (5.4b)) are significantly lower due to the control strategy (C) as compared with strategies (A) and (B).

Figure (5.5) shows in order to control OG addiction with minimal cost in a short time, strategy (C) is better than other strategies. Therefore, the intervention strategy is effective in minimizing the number of exposed and addicted individuals in the population, including the associated costs. Thus, policymakers may choose this integrated strategy for combating addiction.

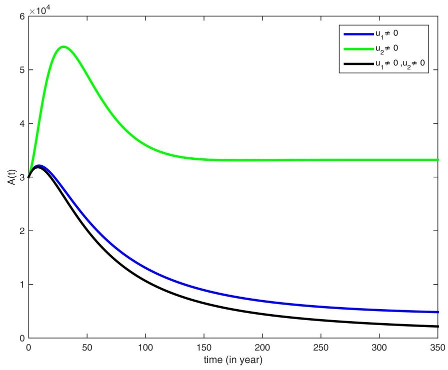


(a) (b)

Figure 5.3: The effect of applying strategy (C).



(a)



(b)

Figure 5.4: Comparison of control strategies.

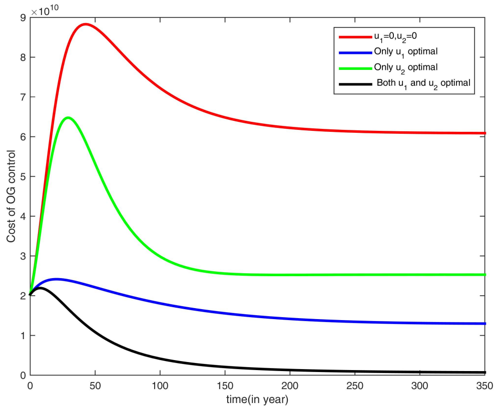


Figure 5.5: Cost comparison with and without control of strategies.

CHAPTER 6
CONCLUSION AND RECOMMENDATIONS

6.1 Conclusion

In this thesis, we considered a deterministic compartmental model to investigate the transmission dynamics of OG addiction with optimal control considering treatment, incomplete recovery rate and complete recovery rate. We established the well-posedness of the model by proving the existence, positivity and boundedness of the solution.

We computed the steady states and the basic reproduction number (R_0) of the model. Our analysis indicates that when $R_0 < 1$, the model has only OG addiction-free equilibrium, which is both locally and globally asymptotically stable and unstable if $R_0 > 1$. Conversely, when $R_0 > 1$, we establish the existence of an endemic equilibrium point that is locally asymptotically stable while OG addiction-free equilibrium is unstable. Using central manifold theory, bifurcation analysis of the model was performed, and the model exhibits forward bifurcation at $R_0 = 1$. We have observed that the addiction outbreak diminishes when R_0 <1, whereas the addiction spreads when R_0 >1. Furthermore, sensitivity analysis of R_0 indicates that the parameters contact rate with addicted individuals(β_1), contact rate with incompletely recovered individuals(β_2), addiction rate(γ), re-addiction rate of incompletely recovered individuals(ω) and incomplete recovery rate of treated individuals(τ) are the most sensitive in our model. In order to eradicate online game addiction from the community, we must reduce the most sensitive parameters contact rate with addicted individuals(β_1), contact rate with incompletely recovered individuals(β_2), addiction rate(γ), re-addiction rate of incompletely recovered individuals(ω) and incomplete recovery rate of treated individuals(τ), which leads to a decrease in the value of R_0, ultimately causing a significant drop in R_0. Additionally, an increase in the parameters treatment rate of addicted individuals(δ), complete recovery rate of incompletely recovered individuals(ε) and complete recovery rate of treated individuals (θ) results in a decrease across all addicted classes

Using optimal control theory, we extended the basic model by including two time dependent control variables: precautionary activity (u_1) and post-cautionary activity (u_2) to minimize the contact rate (β_1) and the re-addiction rate of incompletely recovered individuals (ω) to reduce the number of exposed and addicted individuals from the population with the minimum associated cost. Using the Pontryagin Maximum Principle, we proved the existence of optimal control. From the simulation, we suggest the combined strategy of precautionary activity (u_1) and post-cautionary activity (u_2) is best to reduce the number of exposed and addicted individuals significantly with minimal cost in a short period of time. Researchers have suggested that to prevent people from becoming addicted, we should increase continuous attention to incomplete recovery, guidance and counseling. Additionally, we suggest providing effective treatments, such as cognitive behavioral therapy, counseling, psychotherapy and medication. It is important to ensure that individuals follow the treatment properly without any breaks and encourage those who have not fully recovered to engage in regular activities. Special attention and restrictions on game play should also be imposed. In addition to these measures, conducting an education campaign for susceptible individuals to control addiction through preparatory activities would be the best way to address this issue.

6.2 Recommendations

Our study's findings show that the contact rate with addicted individuals, the contact rate with incompletely recovered individuals, the rate of addiction, the incomplete recovery rate and re-addiction rates are the most influential parameters in the spread of addiction. According to optimal control analysis, the combined control strategy is effective in reducing the spread of addiction and minimizing associated costs. Therefore, we suggest the following recommendations to reduce the burden of online game addiction:

- The stakeholders should educate and supervise families, friends, or relatives of addicted individuals about the negative effects of gaming addiction on mental health, academic performance and social relationships. This helps them monitor addicted individuals' gaming habits and set limits.
- Interested individuals who are popular on media and social platforms integrating with professionals should provide education campaigns to expose individuals to the negative effects of online gaming addiction and teach how to minimize spending time with responsible gaming practices, including setting limits. This helps individuals who are prone to play and at risk of developing addiction develop critical thinking skills to evaluate the impact of online games and make informed decisions about their gaming habits.
- Charitable organizations should do part by setting up support groups for individuals with game addiction and incompletely recovered individuals to provide them with peer support and guidance. Also, connect them with other individuals who understand their experiences and challenges.
- Healthcare professionals should provide effective cognitive-behavioral therapy treatment for individuals with game addiction, including addressing underlying psychological factors such as stress, anxiety, and depression that contribute to game addiction, while individuals who take treatment should follow treatment properly given by professionals without any break.
- Gaming companies should develop apps and software to track gaming time and usage patterns and provide early intervention services to help addicted individuals reduce their gaming time and develop healthier habits.

BIBLIOGRAPHY

[1] Balatif, O., Khajji, B., and Rachik, M. (2020). Mathematical modeling, analysis, and optimal control of abstinence behavior of registration on the electoral lists. Discrete Dynamics in Nature and Society, 2020.

[2] Bhatiasevi, V., Rukumnuaykit, P., and Pholphirul, P. (2023). Online Gaming Addiction and Quality of Life among Early Adolescents in Thailand: An Investigation from a SEM-Neural Network Approach. Human Behavior and Emerging Technologies, 2023.

[3] Byatt-Smith, J. G. (1979). DW Jordan and P. Smith, Nonlinear Ordinary Differential Equations (Oxford Applied Mathematics and Computing Series), 360 pp,£ 12.00 (hard cover). Proceedings of the Edinburgh Mathematical Society, 22(1), 71-71.

[4] Carr, J. (2012). Applications of centre manifold theory (Vol. 35). Springer Science and Business Media.

[5] Castillo-Chavez, C., Blower, S., van den Driessche, P., Kirschner, D., and Yakubu, A. A. (Eds.). (2002). Mathematical approaches for emerging and reemerging infectious diseases: models, methods, and theory (Vol. 126). Springer Science and Business Media.

[6] Castillo-Chavez, C., and Song, B. (2004). Dynamical models of tuberculosis and their applications. Mathematical Biosciences and Engineering, 1(2), 361-404.

[7] Cheneke, K. R. (2023). Optimal control and bifurcation analysis of HIV model. Computational and Mathematical Methods in Medicine, 2023.

[8] D Griffiths, M., J Kuss, D., and L King, D. (2012). Video game addiction: Past, present and future. Current Psychiatry Reviews, 8(4), 308-318.

[9] Diekmann, O., Heesterbeek, J., and Roberts, M. G. (2010). The construction of nextgeneration matrices for compartmental epidemic models. Journal of the Royal Society Interface, 7(47):873– 885.

[10] Duong, X. L., Liaw, S. Y., and Augustin, J. L. P. M. (2020). How has internet addiction been tracked over the last decade? A literature review and 3C paradigm for future research. International Journal of Preventive Medicine, 11.

[11] Endashaw, E. E., Gebru, D. M., and Alemneh, H. T. (2022). Coinfection Dynamics of HBV-HIV/AIDS with Mother-to-Child Transmission and Medical Interventions. Computational and Mathematical Methods in Medicine, 2022.

[12] Endashaw, E. E., and Mekonnen, T. T. (2022). Modeling the effect of vaccination and treatment on the transmission dynamics of hepatitis B virus and HIV/AIDS coinfection. Journal of Applied Mathematics, 2022, 1-27.

[13] Fantaye, A. K., and Birhanu, Z. K. (2022). Chewing Khat Transmission Dynamics: A Mathematical Model and Stability Analysis. Journal of Applied Mathematics, 2022.

[14] Fleming, W. H., and Rishel, R. W. (2012). Deterministic and stochastic optimal control (Vol. 1). Springer Science andBusiness Media.

[15] Gbenga, A. J. (2012). Mathematical modeling and analysis of HIV/AIDS control measures (Doctoral dissertation, University of the Western Cape).

[16] Goode, K., and Vasinda, S. (2021). Videogames and sensory theory: Enchantment in the 21st century. In Disciplinary literacy connections to popular culture in K-12 settings (pp. 162-183). IGI Global.

[17] Guo, Y., and Li, T. (2022). Dynamics and optimal control of an online game addiction model with considering family education. AIMS Mathematics, 7(3), 3745-3770.

[18] Gurmu, E. D., Bole, B. K., and Koya, P. R. (2021). Mathematical modelling of HIV/AIDS transmission dynamics with optimal control strategy. International Journal of Mathematcs and Computer Research, 9(04), 2237-2254.

[19] Haagsma, M. C., Pieterse, M. E., Peters, O., and King, D. L. (2013). How gaming may become a problem: A qualitative analysis of the role of gaming related experiences and cognitions in the development of problematic game behavior. International Journal of Mental Health and Addiction, 11, 441-452.

[20] Heffernan, J. M., Smith, R. J., and Wahl, L. M. (2005). Perspectives on the basic reproductive ratio. Journal of the Royal Society Interface, 2(4), 281-293.

[21] Hyun, G. J., Han, D. H., Lee, Y. S., Kang, K. D., Yoo, S. K., Chung, U. S., and Renshaw, P. F. (2015). Risk factors associated with online game addiction: A hierarchical model. Computers in human behavior, 48, 706-713.

[22] Iggidr, A., Mbang, J., Sallet, G., and Tewa, J. J. (2007, August). Multi-compartment models. In Conference Publications (Vol. 2007, No. Special, pp. 506-519). Conference Publications.

[23] Kardefelt-Winther, D. (2015). A critical account of DSM-5 criteria for internet gaming disorder. Addiction Research and Theory, 23(2), 93-98.

[24] Kaya, A., Türk, N., Batmaz, H., and Griffiths, M. D. (2023). Online gaming addiction and basic psychological needs among adolescents: the mediating roles of meaning in life and responsibility. International Journal of Mental Health and Addiction, 1-25.

[25] Khajji, B., Kouidere, A., Balatif, O., and Rachik, M. (2020). Mathematical modeling, analysis and optimal control of an alcohol drinking model with liver complication. Commun. Math. Biol. Neurosci., 2020, Article-ID.

[26] Király, O., Koncz, P., Griffiths, M. D., and Demetrovics, Z. (2023). Gaming disorder: A summary of its characteristics and aetiology. Comprehensive Psychiatry, 122, 152376.

[27] Korn, G. A., and Korn, T. M. (2000). Mathematical handbook for scientists and engineers: definitions, theorems, and formulas for reference and review. Courier Corporation.

[28] Layek, G. C. (2015). An introduction to dynamical systems and chaos (Vol. 449). New Delhi: Springer.

[29] Lenhart, S., and Workman, J. T. (2007). Optimal control applied to biological models. Chapman and Hall/CRC.

[30] Lenhart, S. M., and Yong, J. (1992). Optimal control for degenerate parabolic equations with logistic growth.

[31] Li, T., and Guo, Y. (2019). Stability and optimal control in a mathematical model of online game addiction. Filomat, 33(17), 5691-5711.

[32] Li, T., and Guo, Y. (2022). Modeling and optimal control of mutated COVID-19 (Delta strain) with imperfect vaccination. Chaos, Solitons and Fractals, 156, 111825.

[33] Li, T., and Guo, Y. (2022). Optimal control strategy of an online game addiction model with incomplete recovery. Journal of Optimization Theory and Applications, 195(3), 780-807.

[34] Lindenberg, K., Kindt, S., and Szász-Janocha, C. (2020). Internet addiction in adolescents: The PROTECT program for evidence-based prevention and treatment. Springer Nature.

[35] Liu, X., Li, J., Zhao, Y., Sun, Y., and Zhang, H. (2023, February). A Kind of Online Game Addictive Treatment Model About Young Person. In International Conference on Emerging Internetworking, Data and Web Technologies (pp. 108-117). Cham: Springer International Publishing.

[36] Martcheva, M. (2015). An introduction to mathematical epidemiology (Vol. 61, pp. 9-31). New York: Springer.

[37] Martončik, M., Ropovik, I., and Adamkovic, M. (2023). Development of Gaming Disorder: Underlying Risk Factors and Complex Temporal Dynamics.

[38] Maulana, D. A. Sensitivity Analysis of Online Game Addiction Model. SITEKIN: Jurnal Sains, Teknologi dan Industri, 20(2), 908-913.

[39] Melesse, D. Y. (2010). Mathematical analysis of an seirs model with multiple latent and infectious stages in periodic and non-periodic environments.

[40] Menon, A. (2019). Coimbatore City Survey on the Best Online Shooting Game.

[41] Miranker, W. L. (1962). Existence, uniqueness and stability of solutions of systems of nonlinear difference-differential equations. Journal of mathematics and mechanics, 101-107.

[42] Mohammad, S., Jan, R. A., Alsaedi, S. L., and Jan, R. (2023). Symptoms, mechanisms, and treatments of video game addiction. Cureus, 15(3).

[43] Muche, H., and Asrese, K. (2022). Prevalence of internet addiction and associated factors among students in an Ethiopian University: A cross-sectional study. Journal of Social Work Practice in the Addictions, 22(2), 143-159.

[44] Nana-Kyere, S., Boateng, F. A., Jonathan, P., Donkor, A., Hoggar, G. K., Titus, B. D. and Adu, I. K. (2022). Global Analysis and optimal control model of COVID-19. Computational and Mathematical Methods in Medicine, 2022.

[45] Nelson, K. E., and Williams, C. M. (Eds.). (2014). Infectious disease epidemiology: theory and practice. Jones and Bartlett Publishers.

[46] Nursalam, N., Iswanti, D. I., Agustiningsih, N., Rohmi, F., Permana, B., and Erwansyah, R. A. (2023). Factors contributing to online game addiction in adolescents: a systematic review. International Journal of Public Health Science (IJPHS), 12(4), 1763.

[47] Pedregal, P. (2004). Introduction to optimization (Vol. 46). New York: Springer.

[48] Perko, L. (2013). Differential equations and dynamical systems (Vol. 7). Springer Science and Business Media.

[49] Pontes, H. M., Kiraly, O., Demetrovics, Z., and Griffiths, M. D. (2014). The conceptualisation and measurement of DSM-5 Internet Gaming Disorder: The development of the IGD-20 Test. PloS one, 9(10), e110137.

[50] Pontes, H. M., and Griffiths, M. D. (2015). Measuring DSM-5 internet gaming disorder: Development and validation of a short psychometric scale. Computers in human behavior, 45, 137-143.

[51] Pontryagin, L. S. (2018). Mathematical theory of optimal processes. Routledge.

[52] Prince, X., Raj, A. V., and Nazini, N. (2023). Impact of Online Gaming Addiction on Social Isolation Among First-Year Sathyabama Undergraduate Students: A Study.

[53] Raffoul, Y. N. (2003). Boundedness In Nonlinear Differential Equations. Nonlinear Studies, 10(4).

[54] Ramesh, K., and Igor, M. (2016). The gaming addiction problem and its economic and social consequences: A comprehensive, dynamic approach. Advanced Engineering Technology and Application, 5(3), 10-18.

[55] Rashid, S., Ashraf, R., and Bonyah, E. (2022). Nonlinear dynamics of the media addiction model using the fractal-fractional derivative technique. Complexity, 2022.

[56] Rodrigues, H. S. F. (2012). Optimal control and numerical optimization applied to epidemiological models (Doctoral dissertation, Universidade de Aveiro (Portugal)).

[57] Rollings, A. (2010). Fundamentals of game design (game design and development series). Prentice-Hall, Inc

[58] Ropovik, I., Martončik, M., Babincak, P., Baník, G., Vargová, L., and Adamkovic, M. (2022). Risk and protective factors for (internet) gaming disorder: A meta-analysis.

[59] Saman, A., Side, S., Pratama, M. I., and Sanusi, W. (2022). Optimal control of the SEIR model of online game addiction using guidance and counseling. Engineering Letters, 30(3), 27-31.

[60] Sastry, S. (2013). Nonlinear systems: analysis, stability, and control (Vol. 10). Springer Science and Business Media.

[61] Side, S., Badwi, N., Pratama, M., Sidjara, S., and Sanusi, W. (2019). A SEIRS model analysis and simulation for dengue fever transmission. International Journal of Scientific and Technology Research., 8(10), 1048-1053.

[62] Side, S., Muzakir, N. A., Pebriani, D., and Utari, S. N. (2021). Model seir kecanduan game online pada siswa di smp negeri 3 makassar. Sainsmat: Jurnal Ilmiah Ilmu Pengetahuan Alam, 9(1), 91-102.

[63] Side, S., Saman, A., Ramadhan, N. R., and Sidjara, S. (2021, November). The Accuracy of the RK-4 and RK-5 Technique as a Numerical Solution to the SEIRS Model of Online Game Addiction on Mathematics Students. In Journal of Physics: Conference Series (Vol. 2123, No. 1, p. 012005). IOP Publishing.

[64] Song, B. (2021). Basic reinfection number and backward bifurcation. Math Biosci Eng, 18(6), 8064-8083.

[65] Sowole, S. O., Sangare, D., Ibrahim, A. A., and Paul, I. A. (2019). On the existence, uniqueness, stability of solution and numerical simulations of a mathematical model for measles disease. Int. J. Adv. Math, 4, 84-111

[66] Stenseng, F., Hygen, B. W., and Wichstrøm, L. (2020). Time spent gaming and psychiatric symptoms in childhood: cross-sectional associations and longitudinal effects. European child and adolescent psychiatry, 29(6), 839-847.

[67] Stevens, M. W., Dorstyn, D., Delfabbro, P. H., and King, D. L. (2021). Global prevalence of gaming disorder: A systematic review and meta-analysis. Australian and New Zealand Journal of Psychiatry, 55(6), 553-568.

[68] Strogatz, S. H. (2018). Nonlinear dynamics and chaos: with applications to physics, biology, chemistry, and engineering. CRC press.

[69] Sublette, V. A., and Mullan, B. (2012). Consequences of play: A systematic review of the effects of online gaming. International journal of mental health and addiction, 10, 3-23.

[70] Suhaimi, A. M., Baharudin, S. K. A. S., Abd Wahab, M. S., Samah, N. H. A., Ariffin, S. A., Elkami, R. M., and Mohammed, A. H. Covid-19 and Physical Activity Levels Among Pharmacy Students: The Impact of Video Gaming.

[71] Sun, R. Q., and Ye, J. H. (2023). The effects of online game addiction on reduced academic achievement motivation among Chinese college students: the mediating role of learning engagement. Frontiers in Psychology, 14, 1185353.

[72] Tilahun, G. T. (2018). Mathematical model for co-infection of pneumonia and typhoid fever disease with optimal control (Doctoral dissertation, JKUAT).

[73] Todorov, E.(2006). Optimal control theory. Bayesian brain: probabilistic approaches to neural coding, 268–298.

[74] Toto, G. (2018). From educational contexts to addictions: The role of technology in teaching methodologies and in prevention as an educational function. Journal of e-Learning and Knowledge Society, 14(2).

[75] Toto, G. A., and Limone, P. (2022). Signs, mechanisms and consequences of Videogame addictions: Educational strategies and Rehabilitation. In teleXbe.

[76] Valcher, M. E. (2002). Positive systems in the behavioral approach: main issues and recent results. Electronic proceedings of MTNS, 2002.

[77] Wang, J. L., Sheng, J. R., and Wang, H. Z. (2019). The association between mobile game addiction and depression, social anxiety, and loneliness. Frontiers in public health, 7, 476493.

[78] Wenzel, H. G., Bakken, I. J., Johansson, A., Götestam, K. G., and Øren, A. (2009). Excessive computer game playing among Norwegian adults: self-reported consequences of playing and association with mental health problems. Psychological reports, 105(3-suppl), 1237-1247.

[79] Yaacob, W. M. Y. W., Zakaria, N. H., and Aji, Z. M. (2021). Identification of factors contributing to online game addiction among adolescents. Journal of Information and Communication Technology, 20(4), 565-597.

[80] Young, K. S. (1999). Internet addiction: Evaluation and treatment. Bmj, 319(Suppl S4).

[81] Zachmanoglou, E. C., and Thoe, D. W. (1986). Introduction to partial differential equations with applications. Courier Corporation.

APPENDIX

Appendix A

```
%%MATLAB code of the system of ode of the model
function ds=oga(t,y)
global A beta1 beta2 gamma omega delta tau
theta epi mu N %parametres of the model
ds=zeros(6,1);
ds(1)=A-(beta1*y(3)+beta2*y(5))*y(1)/N-mu*y(1);
ds(2) =(beta1*y(3)+beta2*y(5))*y(1)/N-(gamma+mu)
*y(2);
ds(3) =gamma*y(2)+omega*y(4)-(mu+ delta)*y(3);
ds(4) =delta*y(3)-(theta+mu+tau)*y(4);
ds(5)=tau*y(4)-(omega+epi+mu)*y(5);
ds(6)=theta*y(4)+epi*y(5)-mu*y(6);
end
```

Appendix B

```
%MATLAB code for stability analysis for R_0>1
close all;clear all;clc
global A beta1 beta2 gamma omega delta tau theta epi
mu N
A=1801.5;beta1=0.553277;beta2=0.36;gamma=0.51;
omega=0.04;delta=0.028;tau=0.02685;theta=0.0612;
epi=0.130325;mu=0.01201;N=150000;;
ts=[0 350];
W=2.5;%line width
         y0=[80000 ,60700,700,2000,300,0];
[t y]=ode45('oga',ts,y0);
plot(t,y(:,1),'LineWidth',W)
xlabel ('time (in year)')
ylabel ('S(t)')
hold on
%%
         y0=[50000 ,44400,40000,14500,1000,100];

[t y]=ode45('oga',ts,y0);
plot(t,y(:,1),'LineWidth',W)
hold on
```

```
%%
         y0=[90000 ,35000,20000,4000,500,500];
[t y]=ode45('oga',ts,y0);
plot(t,y(:,1),'LineWidth',W)
%%
         y0=[10000,45000,60000,30000,10000,4000];
[t y]=ode45('oga',ts,y0);
plot(t,y(:,1),'LineWidth',W)
legend('S(0)=700','S(0)=500','S(0)=895','S(0)=10')
```

Appendix C

```
%MATLAB code for impact of beta1 on Exposed
close all;clear all;clc
format short
global A beta1 beta2 gamma omega delta tau theta
epi mu N
A=1801.5;beta1=0.553277;beta2=0.036;gamma=0.051;
omega=0.04;delta=0.0028;
tau=0.002685;theta=0.0612;epi=0.130325;mu=0.01201;
N=150000;ts=[0 350];
b=linspace(0.1,beta1,4)
for beta1=b;
y0=[69000 ,34000,30000,6000,800,10200];
[t y]=ode45('oga',ts,y0);
plot(t,y(:,2),'LineWidth',2.5)
hold on
legend('\beta_1= 0.1 ','\beta_1=0.2511',
'\beta_1=0.4022','\beta_1=0.553277')
xlabel ('time (in year)')
ylabel ('E(t)')
end
```

Appendix D

```
close all;clear all;clc
%MATLAB codes for bifurcation analysis
A=1801.5;beta1=0.553277;beta2=0.036;gamma=0.051;omega=0.04;
delta=0.0028;tau=0.002685;theta=0.0612;epi=0.130325;
mu=0.01201;
```

```
R1=gamma*((tau+theta+mu)*(omega+mu)*beta1+delta*tau*beta2)
/((gamma+mu)*((delta+mu)*(theta+delta+mu)*(omega+epi+mu)
-omega*delta*tau));
R1
R=1:0.01:2;
R1=0:0.01:1;
%%
fprintf('Value of parameter R0 is %.5f',gamma*((tau+theta+mu)*
(omega+mu)*beta1+delta*tau*beta2)/((gamma+mu)*((delta+mu)*
(theta+delta+mu)*(omega+epi+mu)-omega*delta*tau)))
Lambda = (mu*(1-1./R))/(1); %obtaind from R
Lambd1 =zeros(length(R),1);
Lambd2 =zeros(length(R),1);
h=figure;
plot(R,Lambda,'g','Linewidth',3)
hold on
plot(R,Lambd1,'r--','Linewidth',3)
plot(R1,Lambd2,'b','Linewidth',3)
xlabel('R_0');
ylabel('Force of addiction,\lambda*');
title('Bifurcation diagram for the OG addiction model')
legend('Stable E_1', 'Unstable E_0','Stable E_0')
hold on
set(h,'Units','Inches');
pos=get(h,'Position');
set(h,'PaperPositionMode','Auto','PaperUnits','Inches',
'PaperSize',[pos(3),pos(4)])print(h,'Forwardbifurication',
'-dpdf','-r0')
```

Appendix E

```
%Sensetivity analysis plot for beta1
%(impact of \beta_1  on R_{0})
close all;clear all;clc
beta1=0.1:0.1:0.553277;
beta2=0.036;
gamma=0.051;
omega=0.04;
delta=0.0028;
tau=0.002685;
theta=0.0612;
```

```
epi=0.130325;
mu=0.01201;
R0=gamma*((tau+theta+mu)*(omega+mu).*beta1+delta*tau
*beta2)/((gamma+mu)*((delta+mu)*(theta+delta+mu)
*(omega+epi+mu)-omega*delta*tau));
plot(beta1,R0,'r','linewidth',4)
legend('R_{0}Vs \beta_1  ');
xlabel('\beta_1 ');
ylabel('R_{0}');
title('The impact of \beta_1  on R_{0}');
```

Appendix F

```
close all;clear all;clc
%function dy = OGAWithControlbetat1
% Parameters of the model
a=1801.5;beta1=0.553277;beta2=0.036;gamma=0.051;
omega=0.04;delta=0.028;tau=0.002685;theta=0.0612;
epi=0.130325;mu=0.01201;N=150000;T=350;
% WEIGHT FACTORS
%K1 = 100; K2 = 100; C1 = 90;C2 =90;
K1=2;K2=2;C1=9;C2=9;
% Parameters of the Runge-Kutta (4th order) method
test = -1; deltaError = 0.0001; M = 1000;
t = linspace(0,T,M+1);
h = T/M;
h2 = h/2; h6 = h / 6;
S = zeros(1,M+1); E = zeros(1,M+1);A= zeros(1,M+1);
T = zeros(1,M+1); Qi = zeros(1,M+1);Qp = zeros(1,M+1);
% Initial conditions of the model
S(1) = 69000; E(1) = 34000; A(1) = 30000; T(1) = 6000;Qi(1)=800;
Qp(1)=10200;
%Vectors for system restrictions and control
Lambda1 = zeros(1,M+1); Lambda2 = zeros(1,M+1);
Lambda3 = zeros(1,M+1);Lambda4 = zeros(1,M+1);
Lambda5 = zeros(1,M+1);Lambda6 = zeros(1,M+1);
U3= zeros(1,M+1);
while(test < 0)
oldS = S; oldE = E;oldA = A; oldT = T; oldQi = Qi;
oldQp = Qp;oldLambda1 = Lambda1; oldLambda2 =Lambda2;
oldLambda3 = Lambda3;oldLambda4 = Lambda4;
```

```
oldLambda5= Lambda5;oldLambda6 = Lambda6;oldU3= U3;
for i = 1:M
% Differential equations of the model
% First Runge-Kutta parameter;
aux1 =  (beta1 * A(i)+beta2*Qi(i)) * S(i)/N;
auxS1 = a- aux1 -mu* S(i);
auxE1 = aux1 - (gamma+mu)*E(i);
auxA1 = gamma*E(i)+(1 - U3(i))*omega*Qi(i)-(delta+mu)
*A(i);auxT1 = delta*A(i)-(tau+theta+mu)*T(i);
auxQi1 = tau*T(i)-((1 - U3(i))*omega+epi+mu)*Qi(i);
auxQp1 = epi*Qi(i)+theta*T(i)-mu*Qp(i);
% Second Runge-Kutta parameter
auxU3 = 0.5 * (U3(i) + U3(i+1));
auxS = S(i) + h2 * auxS1; auxE = E(i) + h2 * auxE1;
auxA = A(i)+ h2*auxA1;auxT = T(i) + h2 * auxT1;
auxQi = Qi(i) + h2 * auxQi1;auxQp = Qp(i)+ h2*auxQp1;
aux1 = (beta1 * auxA+beta2*auxQi) * auxS/N;
auxS2 = a- aux1 -mu* auxS;
auxE2 = aux1 - (gamma+mu)*auxE;
auxA2 = gamma*auxE+(1 - auxU3) * omega*auxQi-(delta+mu)
*auxA;auxT2 = delta*auxA-(tau+theta+mu)*auxT;
auxQi2 = tau*auxT-((1 - auxU3) * omega+epi+mu)*auxQi;
auxQp2 = epi*auxQi+theta*auxT-mu*auxQp;
% Threed Runge-Kutta parameter
auxS = S(i) + h2 * auxS2; auxE = E(i) + h2 * auxE2;
auxA = A(i) + h2 * auxA2;auxT = T(i) + h2 * auxT2;
auxQi = Qi(i) + h2 * auxQi2;auxQp = Qp(i)+h2*auxQp2;
aux1 = ( beta1 * auxA+beta2*auxQi) * auxS/N;
auxS3 = a- aux1 -mu* auxS;
auxE3 = aux1 - (gamma+mu)*auxE;
auxA3 = gamma*auxE+(1 - auxU3) * omega*auxQi-
(delta+mu)*auxA;auxT3 = delta*auxA-(tau+theta+mu)*auxT;
auxQi3 = tau*auxT-((1 - auxU3) * omega+epi+mu)*auxQi;
auxQp3 = epi*auxQi+theta*auxT-mu*auxQp;
% Fourth Runge-Kutta parameter
auxS = S(i) + h2 * auxS3; auxE = E(i) + h2 * auxE3;
auxA = A(i) + h2 * auxA3;auxT = T(i) + h2 * auxT3;
auxQi = Qi(i) + h2 * auxQi3;auxQp = Qp(i)+h2*auxQp3;
aux1 = (beta1 * auxA+beta2*auxQi) * auxS/N;
auxS4 =a- aux1 -mu* auxS;
auxE4 = aux1 - (gamma+mu)*auxE;
auxA4 = gamma*auxE+(1 - auxU3) * omega*auxQi-
```

```
(delta+mu)*auxA;auxT4 = delta*auxA-(tau+theta+mu)
*auxT;auxQi4 = tau*auxT-((1 - auxU3) * omega+epi+mu)
*auxQi;auxQp4 = epi*auxQi+theta*auxT-mu*auxQp;
% Runge-Kutta new approximation
S(i+1)= S(i)+h6*(auxS1 + 2 * (auxS2 + auxS3)+auxS4);
E(i+1)= E(i)+h6*(auxE1 + 2 * (auxE2 + auxE3)+auxE4);
A(i+1)= A(i)+h6*(auxA1 + 2 * (auxA2 + auxA3)+auxA4);
T(i+1)= T(i)+h6*(auxT1 + 2 * (auxT2 + auxT3)+auxT4);
Qi(i+1)= Qi(i)+h6*(auxQi1 + 2*(auxQi2+auxQi3)+auxQi4);
Qp(i+1)= Qp(i)+h6*(auxQp1 + 2*(auxQp2+auxQp3)+auxQp4);
end
%Backward Runge-Kutta iterations
for i = 1:M
j = M + 2 - i;
% Differential equations of the model
% First Runge-Kutta parameter
auxU3= 1 - U3(j);
aux1 = (beta1 * A(i)+beta2*Qi(i))/N;
auxLambda11 =Lambda1(j) *( aux1+mu) - Lambda2(j) * aux1;
auxLambda21 =-K1+Lambda2(j) *(gamma+mu)-Lambda3(j)*(gamma);
auxLambda31 =-K2+Lambda1(j) *(beta1)*S(i)/N - Lambda2(j)*...
( beta1)*S(i)/N+Lambda3(j)*(delta+mu)-Lambda4(j)*delta;
auxLambda41 =Lambda4(j) *(tau+theta) - Lambda5(j)
 *tau-Lambda6(j)*theta;
auxLambda51 =Lambda1(j) *beta2*S(i)/N - Lambda2(j) *
 beta2*S(i)/N...
-Lambda3(j)*auxU3 * omega+Lambda5(j)*(auxU3 * omega+epi+mu);
auxLambda61=Lambda6(j)*mu;
% Second Runge-Kutta parameter
auxU3 = 1 - 0.5 * (U3(j) + U3(j-1));
auxS = 0.5 * (S(j) + S(j-1));
auxA = 0.5 * (A(j) + A(j-1));
auxQi = 0.5 * (Qi(j) + Qi(j-1));
aux1 = (beta1 * auxA+beta2*auxQi)/N;
auxLambda1 = Lambda1(j) - h2 * auxLambda11;
auxLambda2 = Lambda2(j) - h2 * auxLambda21;
auxLambda3 = Lambda3(j) - h2 * auxLambda31;
auxLambda4 = Lambda4(j) - h2 * auxLambda41;
auxLambda5 = Lambda5(j) - h2 * auxLambda51;
auxLambda6 = Lambda6(j) - h2 * auxLambda61;
auxLambda12 =auxLambda1 *( aux1+mu) - auxLambda2 * aux1;
auxLambda22 =-K1+auxLambda2 *(gamma+mu)-auxLambda3*(gamma);
```

```
auxLambda32 =-K2+auxLambda1 *(beta1)*auxS/N - auxLambda2...
*(beta1)*auxS/N+auxLambda3*(delta+mu)-auxLambda4*delta;
auxLambda42=auxLambda4*(tau+theta+mu)-auxLambda5*tau...
-auxLambda6*theta;auxLambda52=auxLambda1*beta2*auxS/N-
   auxLambda2* beta2*auxS/N-auxLambda3*auxU3 *
   omega+auxLambda5*(auxU3*omega+epi+mu)-auxLambda6*epi;
auxLambda62=auxLambda6*mu;
% 3rd Runge-Kutta parameter
auxU3 = 1 - 0.5 * (U3(j) + U3(j-1));
auxS = 0.5 * (S(j) + S(j-1));
auxA = 0.5 * (A(j) + A(j-1));
auxQi = 0.5 * (Qi(j) + Qi(j-1));
aux1 = (beta1 * auxA+beta2*auxQi)/N;
auxLambda1 = Lambda1(j) - h2 * auxLambda12;
auxLambda2 = Lambda2(j) - h2 * auxLambda22;
auxLambda3 = Lambda3(j) - h2 * auxLambda32;
auxLambda4 = Lambda4(j) - h2 * auxLambda42;
auxLambda5 = Lambda5(j) - h2 * auxLambda52;
auxLambda6 = Lambda6(j) - h2 * auxLambda62;
auxLambda13 =auxLambda1 *( aux1+mu) - auxLambda2 * aux1;
auxLambda23 =-K1+auxLambda2 *(gamma+mu)-auxLambda3*(gamma);
auxLambda33 =-K2+auxLambda1 *(beta1)*auxS/N - auxLambda2...
*(beta1)*auxS/N+auxLambda3*(delta+mu)-auxLambda4*delta;
auxLambda43 =auxLambda4*(tau+theta)-auxLambda5
*tau-auxLambda6*theta;auxLambda53=auxLambda1*
beta2*auxS/N - auxLambda2* beta2*auxS/N-auxLambda3*auxU3*
omega+auxLambda5*(auxU3*omega+epi+mu)-auxLambda6*epi;
auxLambda63=auxLambda6*mu;
% 4th Runge-Kutta parameter
auxU1 = 1 - 0.5 * (U3(j) + U3(j-1));
auxS = 0.5 * (S(j) + S(j-1));
auxA = 0.5 * (A(j) + A(j-1));
auxQi = 0.5 * (Qi(j) + Qi(j-1));
aux1 = (beta1 * auxA+beta2*auxQi)/N;
auxLambda1 = Lambda1(j) - h2 * auxLambda31;
auxLambda2 = Lambda2(j) - h2 * auxLambda23;
auxLambda3 = Lambda3(j) - h2 * auxLambda33;
auxLambda4 = Lambda4(j) - h2 * auxLambda43;
auxLambda5 = Lambda5(j) - h2 * auxLambda53;
auxLambda6 = Lambda6(j) - h2 * auxLambda63;
auxLambda14 =auxLambda1 *( aux1+mu) - auxLambda2 * aux1;
auxLambda24 =-K1+auxLambda2 *(gamma+mu)-auxLambda3*(gamma);
```

```
auxLambda34 =-K2+auxLambda1 *(beta1)*auxS/N - auxLambda2...
*(beta1)*auxS/N+auxLambda3*(delta+mu)-auxLambda4*delta;
auxLambda44 =auxLambda4*(tau+theta) - auxLambda5 *tau-
auxLambda6*theta;auxLambda54=auxLambda1*beta2*auxS/N-
auxLambda2* beta2*auxS/N-auxLambda3*auxU1 *
omega+auxLambda5*(auxU1 * omega+epi+mu)-auxLambda6*epi;
auxLambda64=auxLambda6*mu;
% Runge-Kutta new approximation
Lambda1(j-1) = Lambda1(j) - h6 * (auxLambda11 + 2 *...
(auxLambda12 + auxLambda13) + auxLambda14);
Lambda2(j-1) = Lambda2(j) - h6 * (auxLambda21 + 2 *...
(auxLambda22 + auxLambda23) + auxLambda24);
Lambda3(j-1) = Lambda3(j) - h6 * (auxLambda31 + 2 *...
(auxLambda32 + auxLambda33) + auxLambda34);
Lambda4(j-1) = Lambda4(j) - h6 * (auxLambda41 + 2 *...
(auxLambda42 + auxLambda43) + auxLambda44);
Lambda5(j-1) = Lambda5(j) - h6 * (auxLambda51 + 2 *...
(auxLambda52 + auxLambda53) + auxLambda54);
Lambda6(j-1) = Lambda6(j) - h6 * (auxLambda61 + 2 *...
(auxLambda62 + auxLambda63) + auxLambda64);
end
% New vector control
for i = 1:M+1
vAux3(i) = omega *Qi(i) * (Lambda3(i) - Lambda5(i))/(C2);
auxU3 = max(0,min([0.9 vAux3(i)]));
U3(i) = 0.5 * (auxU3 + oldU3(i));
end
% Absolute error for convergence
temp1 = deltaError*sum(abs(U3))-sum(abs(oldU3-U3));
temp2 = deltaError*sum(abs(S))-sum(abs(oldS-S));
temp3 = deltaError*sum(abs(E))-sum(abs(oldE-E));
temp4 = deltaError*sum(abs(A))-sum(abs(oldA-A));
temp5 = deltaError*sum(abs(T))-sum(abs(oldT-T));
temp6= deltaError*sum(abs(Qi))-sum(abs(oldQi-Qi));
temp7 = deltaError*sum(abs(Qp))-sum(abs(oldQp-Qp));
temp8=deltaError*sum(abs(Lambda1))-sum(abs(oldLambda1
-Lambda1));
temp9=deltaError*sum(abs(Lambda2))-sum(abs(oldLambda2
-Lambda2));
temp10=deltaError*sum(abs(Lambda3))-sum(abs(oldLambda3
-Lambda3));
temp11=deltaError*sum(abs(Lambda4))-sum(abs(oldLambda4
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-Lambda4));
temp12=deltaError*sum(abs(Lambda5))-sum(abs(oldLambda5
-Lambda5));
temp13=deltaError*sum(abs(Lambda6))-sum(abs(oldLambda6
-Lambda6));
test=min([temp1 temp2 temp3 temp4 temp5 temp6 temp7 temp8
temp9 temp10 temp11 temp12 temp13]);
end
 dw(1,:) = t; dw(2,:) = S; dw(3,:) = E;dw(4,:) = A;
 dw(5,:) = T;dw(6,:) = Qi; dw(7,:) = Qp;dw(8,:) = U3;
 figure(1)
 plot(dw(1,:),dw(3,:),'k','LineWidth',2.5)
 legend('u_1=0 ,u_2\neq 0')
 xlabel ('time (in year)')
 ylabel ('E(t)')
 %%
 figure(2)
 plot(dw(1,:),dw(4,:),'k','LineWidth',2.5)
 legend('u_1= 0 ,u_2\neq 0')
 xlabel ('time (in year)')
 ylabel ('A(t)')
```